Forelesning 16/11-2016

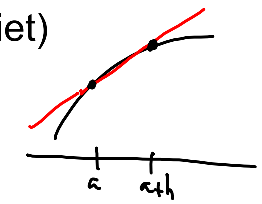
Sist: feilanalyse numerisk derivasjon (kap 11 i kompendiet)

Idag:

- Fortsetter der vi slapp
- Feilanalyse numerisk integrasjon (kap 12 i kompendiet)

Fredag: numerisk løsning av ligninger

Feilanalyse numerisk derivasjon (kap 11 i kompendiet)



Fra Taylor

$$\left| f'(a) - \frac{f(a+h)-f(a)}{h} \right| = \frac{h}{2} \left| f''(\xi) \right|$$

$h > 0$

$\xi = \xi(a) \in (a, a+h)$

<u>Eks</u> $f(x) = \sin x \qquad a = 0.5$

h	observert feil
10^{-1}	$2.5 \cdot 10^{-2}$
10^{-2}	$2.4 \cdot 10^{-3}$
10^{-3}	$2{,}397 \cdot 10^{-4}$
$\vdots$	$\vdots$
10^{-6}	$2{,}397 \cdot 10^{-7}$

„Teoretisk" feil $= \frac{h}{2} \cdot \left| f''(\xi) \right|$

$$\approx \frac{h}{2} \cdot \left| f''(a) \right|$$

$$\approx h \cdot 0.2397$$

siden $\left| f''(a) \right| = \left| \sin(0.5) \right| = 0.4794$

....

Stemmer bra!

Hva skjer om vi fortsetter med mindre h?

h	$(f(a+h)-f(a))/h$	$E(f;a,h)$
10^{-7}	0. 8775825372	2.5×10^{-8}
10^{-8}	0.8775825622	-2.9×10^{-10}
10^{-9}	0.8775825622	-2.9×10^{-10}
10^{-11}	0.8775813409	1.2×10^{-6}
10^{-14}	0.8770761895	5.1×10^{-4}
10^{-15}	0.8881784197	-1.1×10^{-2}
10^{-16}	1.110223025	-2.3×10^{-1}
10^{-17}	0.000000000	8.8×10^{-1}

Problemet er avrundingsfeil!

Avrundingsfeil numerisk derivasjon

Forklaring et avrundings feil i $f(a)$ og $f(a+h)$

På datamaskin får vi nærmeste flyttall $\overline{f(a)}$ og $\overline{f(a+h)}$

$\overline{f(a)} = f(a)(1+\varepsilon_1)$ hvor ε_1 er relativ feil $|\varepsilon_1| \approx 6\times 10^{-16}$

(64 bits flyttall)

$\overline{f(a+h)} = f(a+h)(1+\varepsilon_2)$ hvor ε_2 relativ feil $|\varepsilon_2| \approx 6\times 10^{-16}$

Setter inn i feiluttrykket

$$f'(a) - \frac{\overline{f(a+h)} - \overline{f(a)}}{h} = f'(a) - \frac{f(a+h)(1+\varepsilon_1) - f(a)(1+\varepsilon_2)}{h}$$

$$= \underbrace{f'(a) - \frac{f(a+h) - f(a)}{h}}_{\text{Trunkeringsfeil}} - \underbrace{\frac{f(a+h)\varepsilon_2 - f(a)\varepsilon_1}{h}}_{\text{Avrundingsfeil}}$$

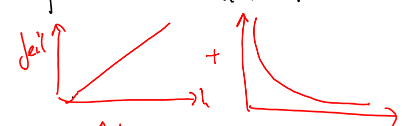
$$= -\frac{h}{2} f''(\xi) - \frac{f(a+h)\varepsilon_2 - f(a)\varepsilon_1}{h}$$

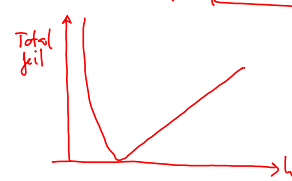
pyntet uttrykk

$\varepsilon^* \approx 7\cdot 10^{-17}$ max avrundings-feil

$$\left| f'(a) - \frac{\overline{f(a+h)} - \overline{f(a)}}{h} \right| \approx \frac{h}{2}\left|f''(a)\right| + \frac{2\varepsilon^*}{h}\left|f(a)\right|$$

Grafisk





Finnes en optimal h?

I prinsippet er feilen (i vår enkle modell)

$E(h) = C_1 h + C_2 \cdot \frac{1}{h}$

Finn minimum i h som gir $E'(h) = 0$

Løser for h og finner

$$h^* = \sqrt{\frac{C_2}{C_1}} = 2\sqrt{\frac{\varepsilon^* |f(a)|}{|f''(a)|}}$$ (optimal steglengde h)

Stemmer dette?

Med $f(x) = \sin x$ $a = 0.5$

$$h^* = 2\cdot\sqrt{\frac{7\cdot 10^{-17}\cdot \cancel{|f(a)|}}{\cancel{|f''(a)|}}} \approx 10^{-8}$$ „optimal" h

Gir avrundingsfeil

$$\frac{2\varepsilon^*}{h^*}|\sin(0.5)| \approx 2\cdot 10^{-9}$$

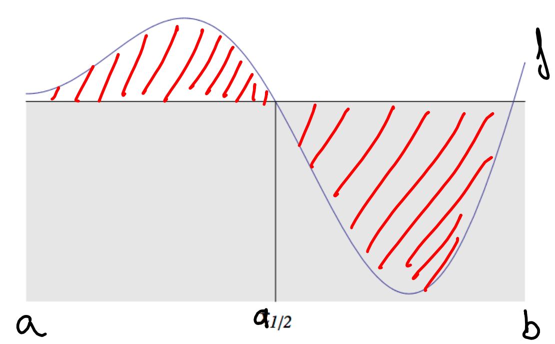
og trunkeringsfeil

$$\frac{h^*}{2}|\sin 0.5| \approx 4\cdot 10^{-9}$$

Feilanalyse numerisk integrasjon (12 i komp)

Lokal feilanalyse for midtpunktsmetoden (ett intervall)

$$\int_a^b f(x)\,dx \approx (b-a)\, f(a_{\frac{1}{2}})$$



Feilen er

$$\int_a^b f(x)\,dx - (b-a)\, f(a_{\frac{1}{2}})$$

Analyse ved 1. ordens Taylor-tilnærming rundt $x = a_{\frac{1}{2}}$

$$f(x) = \underbrace{f(a_{\frac{1}{2}}) + (x - a_{\frac{1}{2}}) f'(a_{\frac{1}{2}})}_{\text{Lineær tilnærming}} + \frac{1}{2}(x - a_{\frac{1}{2}})^2 f''(\xi) \quad \text{hvor}$$

$\xi = \xi(x) \in (a_{\frac{1}{2}}, x)$

Integrerer på begge sider

$$\int_a^b f(x)\,dx = \int_a^b f(a_{\frac{1}{2}})\,dx + \int_a^b (x - a_{\frac{1}{2}}) f'(a_{\frac{1}{2}})\,dx + \int_a^b \frac{1}{2}(x - a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

$\xi = \xi(x) \in (a_{\frac{1}{2}}, x)$

$$= \underbrace{(b-a)\, f(a_{\frac{1}{2}})}_{\text{midtpunktsmetoden}} + 0 + \frac{1}{2}\int_a^b (x - a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

pynter på uttrykket:

$$\int_a^b f(x)\,dx - (b-a)\, f(a_{\frac{1}{2}}) = \frac{1}{2}\int_a^b (x - a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

Dette er feilen. Litt upraktisk vil finne en øvre grense for tallverdi av feilen

$$\left| \int_a^b f(x)\,dx - (b-a)\, f(a_{\frac{1}{2}}) \right| = \left| \frac{1}{2} \int_a^b (x - a_{\frac{1}{2}})^2 f''(\xi)\,dx \right|$$

Forklaring

Trekantulikhet

$$\leq \frac{1}{2} \int_a^b |x - a_{\frac{1}{2}}|^2 \, |f''(\xi)|\, dx$$

$$= \frac{1}{2} \int_a^b (x - a_{\frac{1}{2}})^2 \, |f''(\xi)|\, dx$$

Tar maksverdi

$$\leq \frac{1}{2} \int_a^b (x - a_{\frac{1}{2}})^2 \cdot \underbrace{\max_{z \in [a,b]} |f''(z)|}_{\text{konstant}} \, dx$$

$$= \max_{z \in [a,b]} |f''(z)| \cdot \frac{1}{2} \int_a^b (x - a_{\frac{1}{2}})^2 \, dx$$

$a_{\frac{1}{2}} = \frac{a+b}{2}$

$$= \max_{z \in [a,b]} |f''(z)| \cdot \frac{1}{2} \cdot \frac{(b-a)^3}{12}$$

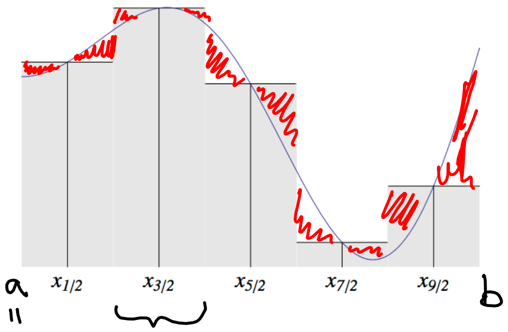
$$= \max_{z \in [a,b]} |f''(z)| \cdot \frac{(b-a)^3}{24}$$

Altså: Dette er en øvre grense for hvor stor feilen i midtpunktsmetoden kan bli for ett intervall

Global feilanalyse for midtpunktsmetoden

$$\int_a^b f(x)\,dx \approx I_{mid}(h) = h\sum_{i=1}^{n} f(x_{i-1/2})$$

$$x_{i-1/2} = (x_{i-1}+x_i)/2 = a + (i-1/2)h.$$



Feil $= \left| \int_a^b f(x)\,dx - h\sum_{i=1}^{n} f(x_{i-\frac{1}{2}}) \right|$

$$= \left| \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - h\cdot f(x_{i-\frac{1}{2}}) \right) \right|$$

Trekantulikhet $\quad \le \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - h f(x_{i-\frac{1}{2}}) \right|$

Forrige resultat $\quad \le \sum_{i=1}^{n} \max_{z\in[x_{i-1},x_i]} |f''(z)| \cdot \frac{h^3}{24}$

Estimerer $f''(z)$ over hele $[a,b]$ $\quad \le \frac{h^3}{24} \cdot \sum_{i=1}^{n} \max_{z\in[a,b]} |f''(z)|$

$$= \frac{h^3}{24} \cdot n \cdot \max_{z\in[a,b]} |f''(z)|$$

$h\cdot n = \frac{b-a}{\cancel{n}}\cdot\cancel{n}$ $\quad = \frac{h^2}{24} \cdot h\cdot n \cdot \max_{z\in[a,b]} |f''(z)|$

$$= \frac{h^2}{24}(b-a)\max_{z\in[a,b]}|f''(z)|$$

Dette er en øvre grense for feilen som funksjon av h, b, a og f

Eksempel:

$$\int_0^1 \cos x\, dx = \sin 1 \approx 0.8414709848$$

h	$I_{\text{mid}}(h)$	Error
0.500000	0.85030065	-8.8×10^{-3}
0.250000	0.84366632	-2.2×10^{-3}
0.125000	0.84201907	-5.5×10^{-4}
0.062500	0.84160796	-1.4×10^{-4}
0.031250	0.84150523	-3.4×10^{-5}
0.015625	0.84147954	-8.6×10^{-6}
0.007813	0.84147312	-2.1×10^{-6}
0.003906	0.84147152	-5.3×10^{-7}
0.001953	0.84147112	-1.3×10^{-7}
0.000977	0.84147102	-3.3×10^{-8}

Estimering av steglengde h

Feilen : $\frac{h^2}{24} \max |f''(z)| (b-a)$ *

Hvis vi feks ønsker at denne skal være mindre enn en ε:

Velg h slik at

$$\frac{h^2}{24} \max_z |f''(z)| (b-a) < \varepsilon$$

I eksemplet med $f(x) = \cos x$, vet at $|f''(x)| \leq 1$

Løser for h når $\varepsilon = 10^{-10}$

$$\frac{h^2}{24} \cdot 1 \cdot 1 < \varepsilon$$

og får $h \leq \sqrt{24 \cdot \varepsilon} \approx \underline{5 \cdot 10^{-5}}$