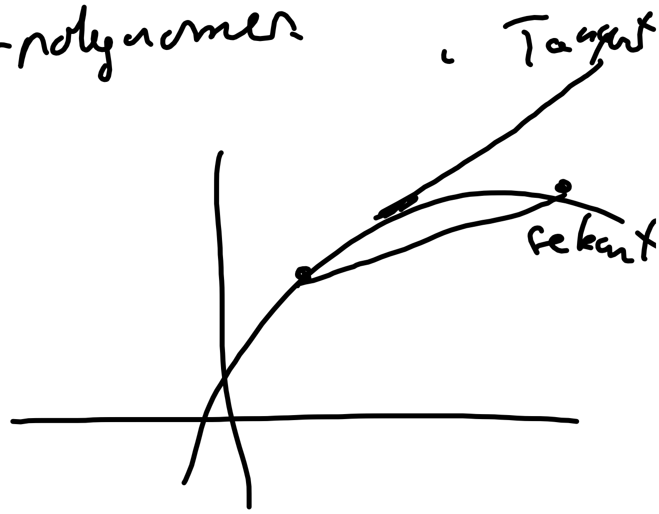
Polynomer er kent!

For å jobbe med generelle funksjoner kan vi tilnærme med polynom og jobbe med det i steden.

To utfordringer:

1. Hvordan finne polynomet?
2. Passe på feilen

En mulighet: Taylorpolynomer



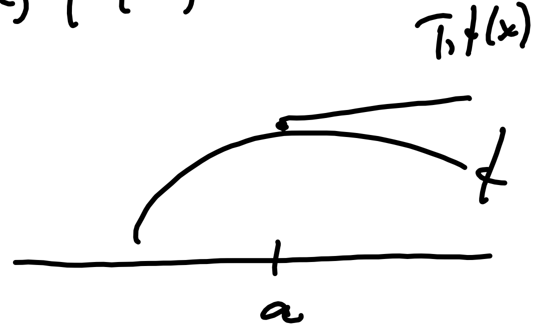
Taylor-polynomer

Generalisering av tangenten til f i a. $c_0 + c_1 x$

$P_1(x) = T_1 f(x) = f(a) + (x-a) f'(a)$

$T_1 f(a) = f(a)$

$(T_1 f)'(a) = f'(a)$



Hver gang vi øker graden kan vi matche en ny derivert.

Grad 2.

Polynom av grad 2.

$$P_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2$$

Ønsker

$$P_2(a) = f(a), \quad P_2'(a) = f'(a)$$

$$P_2''(a) = f''(a)$$

$$P_2(a) = c_0 = f(a)$$

$$P_2'(x) = c_1 + 2c_2(x-a)$$

$$P_2'(a) = c_1 = f'(a)$$

$$P_2''(x) = 2c_2 = f''(a), \quad c_2 = \frac{f''(a)}{2}$$

$$P_2(x) = f(a) + (x-a)f'(a) + (x-a)^2 \frac{f''(a)}{2}$$

Taylorpolynom av grad n

Polynomet

$$T_n f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

tilfredstiller betingelsene

$$T_n f(a) = f(a), \ldots, (T_n f)^{(k)}(a) = f^{(k)}(a)$$

$k = 0, 1, \ldots, n.$

Taylor polynomet til e^x:

$$T_n f(x) = f(a) + f'(a)(x-a) + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

$f(x) = e^x$, $a = 0$, $f^{(k)}(x) = e^x$, $k \geq 0$

$f^{(k)}(0) = 1$

$$T_n f(x) = 1 + 1\cdot(x-0) + \cdots + 1\,\frac{(x-0)^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

$f(x) = \sin x \, , \quad a = 0$

$T_n f(x) = f(a) + (x-a) f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a)$

$f(0) = 0$

$f'(x) = \cos x \, , \quad f'(0) = 1$

$f''(x) = -\sin x \, , \quad f''(0) = 0$

$f'''(x) = -\cos x \, , \quad f'''(0) = -1$

$f^{iv}(x) = \sin x \, , \quad f^{iv}(0) = 0$

$T_{2n-1} \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} = \cos x + i \sin x \quad !!!$$

Restleddet

Vi vet at

$$f(b) - f(a) = \int_a^b f'(t)\,dt$$

Sett $b = x$

$$f(x) = \underbrace{f(a)}_{T_0 f(x)} + \underbrace{\int_a^x f'(t)\,dt}_{\text{Feilen } R_0 f(x)}$$

For å få fram feilen i $T_1 f(x)$ så delvis integrerer vi

$$f(b) = f(a) + \int_a^b 1\cdot f'(t)\,dt$$

Delvis integrasjon: $\int u \cdot v' = u \cdot v - \int u' v$

$u = f'(t)$, $v' = 1$,

$u' = f''(t)$, $v = t - b$

$$\int_a^b 1 \cdot f'(t)\,dt = \left[f'(a)(t-b) \right]_a^b - \int_a^b f''(t)(t-b)\,dt$$

$$= 0 - f'(a)(a-b) + \int_a^b f''(t)(b-t)\,dt$$

$$= f'(a)(b-a) + \int_a^b f''(t)(b-t)\,dt$$

Innsatt i $f(b) = f(a) + \int_a^b f'(t)\,dt$

$$f(\overset{x}{b}) = f(a) + (\overset{x}{b} - a) f'(a) + \int_a^b f''(t)(b-t)\,dt$$

$$f(x) = \underbrace{f(a) + (x-a) f'(a)}_{T_1 f(x)} + \underbrace{\int_a^x f''(t)(x-t)\,dt}_{R_1 f(x)}$$