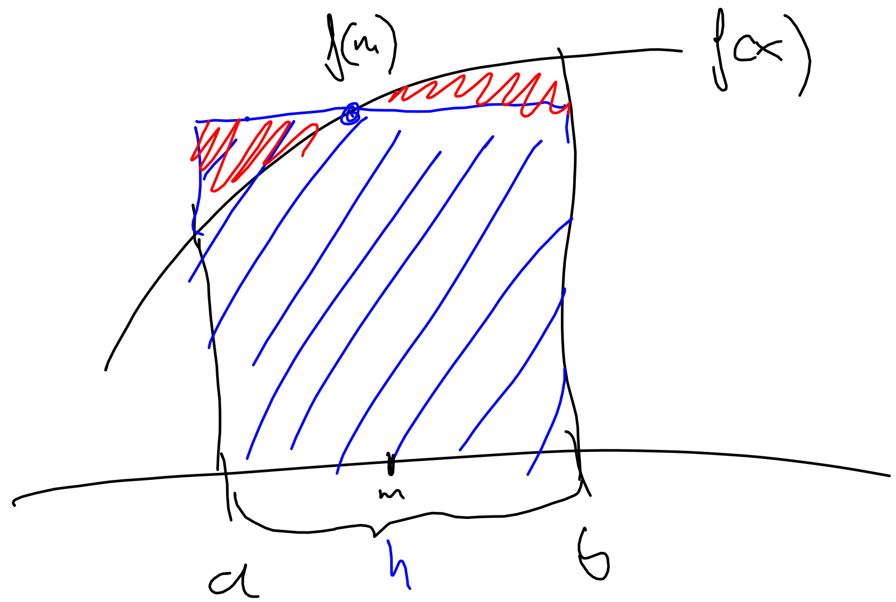
# MAT-INF1100 - Forelesning 27/10-2017

IDAG: FEILANALYSE FOR NUMERISK INTEGRASJON



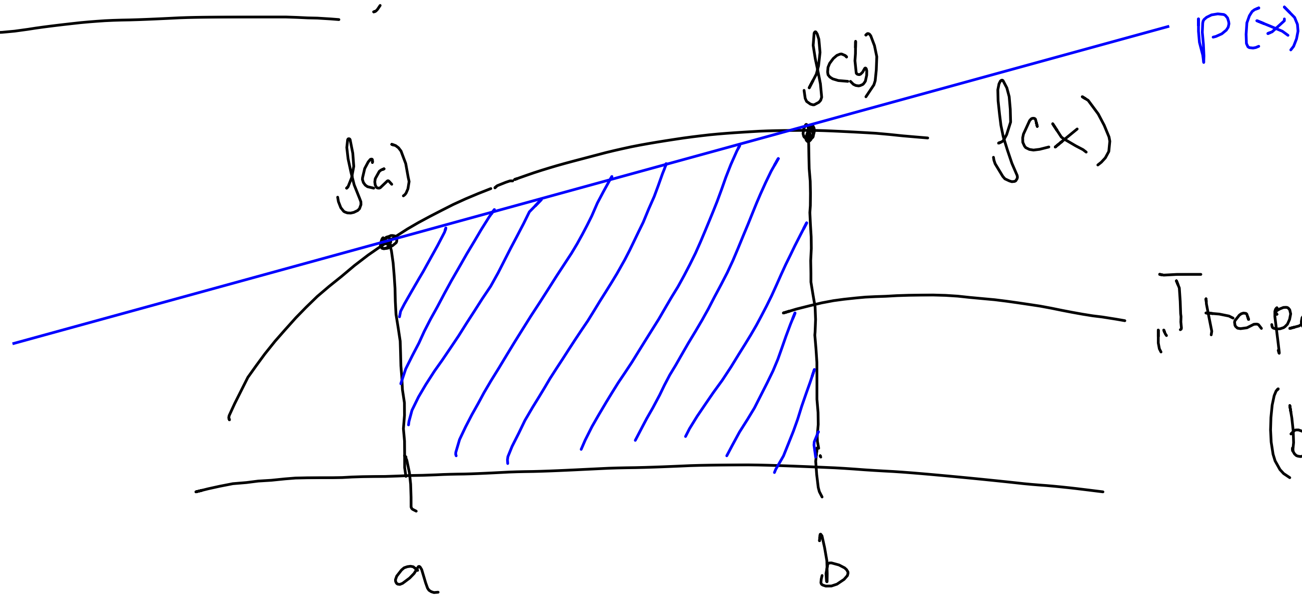

$$\int_a^b f(x)\,dx \approx h \cdot f(m)$$

$h = b - a$

Sist: **Trapes-metoden**

Ide: tilnærme $f$ med lineært polynom $p(x)$ som interpolerer $f(x)$ i to punkter

På ett intervall:



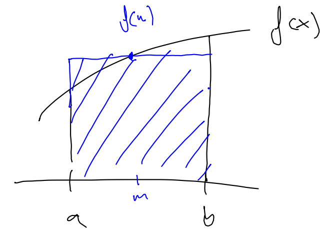

„Trapes" med areal

$$(b-a)\cdot\frac{f(a)+f(b)}{2}$$

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx = (b-a)\cdot\frac{f(a)+f(b)}{2}$$

# FEIL-ANALYSE FOR MIDTPUNKTSMETODEN

$$\int_a^b f(x)\,dx \approx h\cdot f(m)$$




Hva blir feilen

$$\int_a^b f(x)\,dx - h\cdot f(m) \quad ?$$

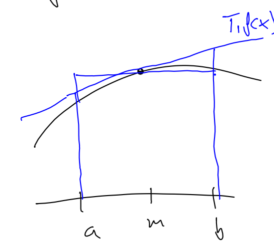
$m = \frac{a+b}{2}$

$h = b-a$

Vi bruker Taylor-polynomer for å analysere feilen.

$$\boxed{f(x) = T_1 f(x) + R_1 f(x)}$$

$T_1 f(x) = f(m) + f'(m)\cdot(x-m)$




Altså er $f(x) \approx T_1 f(x)$ og da er

$$\int_a^b f(x)\,dx \approx \int_a^b T_1 f(x)\,dx$$

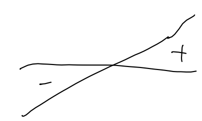
La oss regne ut $\int_a^b T_1 f(x)\,dx$

$$\int_a^b T_1 f(x)\,dx = \int_a^b f(m) + f'(m)(x-m)\,dx$$

$$= \left[f(m)\cdot x\right]_a^b + \underbrace{\left[\frac{f'(m)\cdot(x-m)^2}{2}\right]_a^b}_{=0}$$

$$= (b-a)\cdot f(m) = h\cdot f(m)$$

aka midtpunktsmetoden!




Vet at $\boxed{f(x) - T_1 f(x) = R_1 f(x)}$

$R_1 f(x) = \frac{1}{2}(x-m)^2\cdot f''(\xi)$ hvor $\xi \in (m,x)$

$$\int_a^b f(x)\,dx - \int_a^b T_1 f(x)\,dx = \int_a^b R_1 f(x)\,dx$$

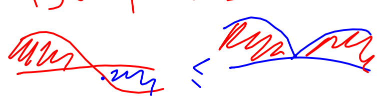
$$\underbrace{\int_a^b f(x)\,dx - h\cdot f(m)}_{\text{Feilen i midtpunktsmetoden}} = \frac{1}{2}\int_a^b (x-m)^2\cdot f''(\xi(x))\,dx$$

Har funnet et uttrykk for feilen, men litt upraktisk

La oss forsøke å finne en øvre grense for tallverdien til feilen

$$\left|\int_a^b f(x)\,dx - h\cdot f(m)\right| = \left|\frac{1}{2}\int_a^b (x-m)^2\cdot f''(\xi(x))\,dx\right|$$

$\left|\int g\,dx\right| \leq \int |g|\,dx$



Trekantulikheten

$$\leq \frac{1}{2}\int_a^b \left|\overbrace{(x-m)^2}^{\geq 0} f''(\xi(x))\right| dx$$

$$= \frac{1}{2}\int_a^b (x-m)^2 \left|f''(\xi(x))\right| dx$$

$\left|f''(\xi(x))\right| \leq \underbrace{\max_{c\in(a,b)} \left|f''(c)\right|}_{\text{konstant}}$

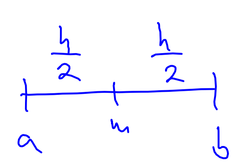
$$\leq \frac{1}{2}\int_a^b (x-m)^2 \cdot \max_{c\in(a,b)}\left|f''(c)\right| dx$$

$$= \max_{c\in(a,b)}\left|f''(c)\right| \cdot \frac{1}{2}\underbrace{\int_a^b (x-m)^2\,dx}$$

$$= \left[\frac{1}{3}(x-m)^3\right]_a^b = \frac{1}{3}\left[(b-m)^3-(a-m)^3\right]$$

$$= \frac{1}{3}\left[\left(\frac{h}{2}\right)^3-\left(-\frac{h}{2}\right)^3\right]$$

$$= \frac{h^3}{12}$$





Altså

$$\left|\int_a^b f(x)\,dx - h\cdot f(m)\right| \leq \max_{c\in(a,b)}\left|f''(c)\right|\cdot\frac{h^3}{24}$$

Øvregrense for feilen i midtpunktsmetoden!

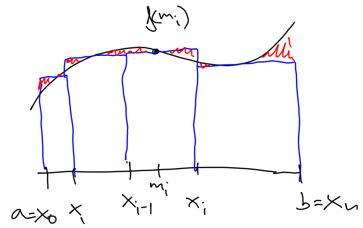
## FEILANALYSE FOR MIDTPUNKTSMETODEN FOR FLERE INTERVALLER

Deler opp $[a,b]$ i $n$ biter med lengde $h = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f(m_i)$$

hvor $x_i = a + i \cdot h$

$m_i = \frac{x_{i-1} + x_i}{2}$




For å estimere feilen bruker vi forrige resultat.

$$|\text{Feil}| = \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} h \cdot f(m_i) \right| = \left| \sum_{i=1}^{n} \left( \int_{x_{i-1}}^{x_i} f(x)\,dx - h \cdot f(m_i) \right) \right|$$

Trekant-ulikhet

$$\leq \sum_{i=1}^{n} \underbrace{\left| \int_{x_{i-1}}^{x_i} f(x)\,dx - h \cdot f(m_i) \right|}_{\text{forrige estimat}}$$

$$\leq \sum_{i=1}^{n} \max_{c \in (x_{i-1}, x_i)} |f''(c)| \cdot \frac{h^3}{24}$$

$$\leq \sum_{i=1}^{n} \max_{c \in (a,b)} |f''(c)| \cdot \frac{h^3}{24}$$

$$= \max_{c \in (a,b)} |f''(c)| \cdot \sum_{i=1}^{n} \frac{h^3}{24}$$

$$= \max_{c \in (a,b)} |f''(c)| \cdot \frac{n \cdot h^3}{24}$$

$$n \cdot h = \cancel{n} \frac{(b-a)}{\cancel{n}} = b-a$$

$$= \max_{c \in (a,b)} |f''(c)| \cdot \frac{(b-a) \cdot h^2}{24}$$

Dette er en øvre grense for feilen som funksjon av $h$ (og $a, b$, og $f$)

$\int_0^1 \cos x \, dx$ med midtpunktsmetoden

h Imid(h) Feilestimat

$$\max_{c \in (a,b)} |f''(c)| \cdot \frac{(b-a) \cdot h^2}{24} \leq \frac{1 \cdot 1 \cdot h^2}{24} = \frac{h^2}{24}$$

```
it=0, h = 1.0000000000, E_mid=3.6e-02, E_est=4.2e-02
it=1, h = 0.5000000000, E_mid=8.8e-03, E_est=1.0e-02
it=2, h = 0.2500000000, E_mid=2.2e-03, E_est=2.6e-03
it=3, h = 0.1250000000, E_mid=5.5e-04, E_est=6.5e-04
it=4, h = 0.0625000000, E_mid=1.4e-04, E_est=1.6e-04
it=5, h = 0.0312500000, E_mid=3.4e-05, E_est=4.1e-05
it=6, h = 0.0156250000, E_mid=8.6e-06, E_est=1.0e-05
it=7, h = 0.0078125000, E_mid=2.1e-06, E_est=2.5e-06
it=8, h = 0.0039062500, E_mid=5.3e-07, E_est=6.4e-07
it=9, h = 0.0019531250, E_mid=1.3e-07, E_est=1.6e-07
it=10, h = 0.0009765625, E_mid=3.3e-08, E_est=4.0e-08
it=11, h = 0.0004882812, E_mid=8.4e-09, E_est=9.9e-09
it=12, h = 0.0002441406, E_mid=2.1e-09, E_est=2.5e-09
it=13, h = 0.0001220703, E_mid=5.2e-10, E_est=6.2e-10
it=14, h = 0.0000610352, E_mid=1.3e-10, E_est=1.6e-10
it=15, h = 0.0000305176, E_mid=3.3e-11, E_est=3.9e-11
```

## Estimere steglengde

Vet at feilen er mindre enn

$$\frac{h^2}{24} \cdot \max_{c \in (a,b)} |f''(c)| \cdot (b-a)$$

Hvis vi f.eks ønsker at feilen skal være mindre enn $10^{-10}$, så kan vi finne en passende $h$.

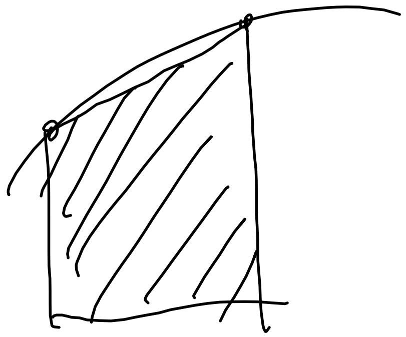
For $\int_0^1 \cos x \, dx$ : Finn $h$ slik at

$$\frac{h^2}{24} \cdot 1 \cdot 1 < 10^{-10}$$

Får da $h \le \sqrt{24 \cdot 10^{-10}} \approx \underline{5 \cdot 10^{-5}}$

## Feil i andre metoder

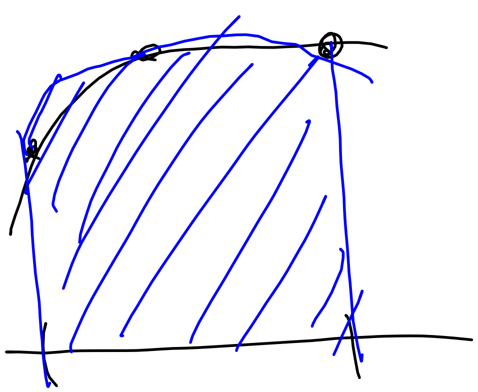
Trapes-metoden (lineær tilnærming til f)



$$\left| \int_a^b f(x)\,dx - I_{\text{trapes}}(h) \right| \le \frac{h^2}{12} \cdot (b-a) \max_{c \in [a,b]} |f''(c)|$$

(litt høyere feil en midtpunktsmetoden)

Simpsons metode (kvadratisk tilnærming)



$$\left| \int_a^b f(x)\,dx - I_{\text{Simpson}}(h) \right| \le \frac{h^4}{180} (b-a) \max_{c \in [a,b]} |f''''(c)|$$

(vesentlig lavere feil)

$$\int_0^1 \cos x \, dx$$

Midpoint Trapes Simpson

```
it=0, h = 1.0000000000, E_mid=3.6e-02, E_trap=7.1e-02, E_simp= 3.3e-01
it=1, h = 0.5000000000, E_mid=8.8e-03, E_trap=1.8e-02, E_simp= 3.0e-04
it=2, h = 0.2500000000, E_mid=2.2e-03, E_trap=4.4e-03, E_simp= 1.8e-05
it=3, h = 0.1250000000, E_mid=5.5e-04, E_trap=1.1e-03, E_simp= 1.1e-06
it=4, h = 0.0625000000, E_mid=1.4e-04, E_trap=2.7e-04, E_simp= 7.1e-08
it=5, h = 0.0312500000, E_mid=3.4e-05, E_trap=6.8e-05, E_simp= 4.5e-09
it=6, h = 0.0156250000, E_mid=8.6e-06, E_trap=1.7e-05, E_simp= 2.8e-10
it=7, h = 0.0078125000, E_mid=2.1e-06, E_trap=4.3e-06, E_simp= 1.7e-11
it=8, h = 0.0039062500, E_mid=5.3e-07, E_trap=1.1e-06, E_simp= 1.1e-12
it=9, h = 0.0019531250, E_mid=1.3e-07, E_trap=2.7e-07, E_simp= 6.8e-14
it=10, h = 0.0009765625, E_mid=3.3e-08, E_trap=6.7e-08, E_simp= 3.4e-15
it=11, h = 0.0004882812, E_mid=8.4e-09, E_trap=1.7e-08, E_simp= 6.7e-16
it=12, h = 0.0002441406, E_mid=2.1e-09, E_trap=4.2e-09, E_simp= 2.3e-15
it=13, h = 0.0001220703, E_mid=5.2e-10, E_trap=1.0e-09, E_simp= 7.8e-16
it=14, h = 0.0000610352, E_mid=1.3e-10, E_trap=2.6e-10, E_simp= 1.4e-15
it=15, h = 0.0000305176, E_mid=3.3e-11, E_trap=6.5e-11, E_simp= 1.6e-15
```

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