Prøveeksamen 2020

Del I

Oppgave 1

$f(x) = \sin(\sin x)$ $\qquad$ $f(0) = \sin(0) = 0$

$f'(x) = \cos(\sin x)\cos(x)$ $\qquad$ $f'(0) = \cos(0)\cos(0) = 1$

$$T_1 f(x) = f(0) + f'(0)(x-0)$$

$$= 0 + 1 \cdot x = \underline{\underline{x}}$$

Alternativ A

Oppgave 2

$y'' - 4y' + 4y = 0$ $\qquad$ $y(0) = 1$ $\quad$ $y'(0) = 1$

Karakteristisk ligning:

$r^2 - 4r + 4 = 0 \Rightarrow r = \frac{4 \pm \sqrt{16 - 4 \cdot 4}}{2} = 2$

Vi har en dobbeltrot, slik at den generelle løsningen er $y(x) = Ce^{2x} + Dxe^{2x}$.

Da er $y'(x) = 2Ce^{2x} + D(1+2x)e^{2x}$

$y(0) = 1$: $\quad C = 1$

$y'(0) = 1$: $\quad 2C + D = 1 \Rightarrow D = 1 - 2C = 1 - 2 = -1$

$$\underline{\underline{y(x) = e^{2x} - xe^{2x}}}$$

Alternativ B

Oppgave 3

$$x^2 y' y^2 = 2x$$

$$y^2 y' = \frac{2}{x}$$

$$\int y^2 \frac{dy}{dx} dx = \int \frac{2}{x} dx$$

$$\frac{1}{3} y^3 = 2\ln|x| + C$$

$$y^3 = 6\ln|x| + C$$

$$y = (6\ln|x| + C)^{\frac{1}{3}}$$

Ser at med $C=0$, og x positiv, så sammenfaller dette med alternativ A.

Oppgave 4

$f(x) = x^2 - 2$ $\qquad x_1 = 1 \qquad x_2 = 2$

$f(x_1) = f(1) = -1 \qquad f(x_2) = f(2) = 2$

$$X_3 = X_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 2 - \frac{2-1}{2-(-1)} 2 = 2 - \frac{1}{3} \cdot 2$$

$$= 2 - \frac{2}{3} = \underline{\underline{\frac{4}{3}}}$$

Alternativ D

Oppgave 5

$$\int_0^2 e^{x^2}\,dx \qquad N=4 \text{ delintervaller}$$

$$h=\frac{b-a}{N}=\frac{2}{4}=\frac{1}{2}$$

$$\int_0^2 f(x)\,dx=\frac{1}{2}\left(\frac{f(0)+f(2)}{2}+f\left(\tfrac{1}{2}\right)+f(1)+f\left(\tfrac{3}{2}\right)\right)$$

$$=\frac{1}{2}\left(\frac{1+e^4}{2}+e^{\frac{1}{4}}+e+e^{\frac{9}{4}}\right)$$

$$\approx \underline{\underline{20.6446}}$$

Alternativ E

Del 2

Oppgave 1

$P_k: \; f^{(k)}(x) = (x+k)e^x$

a) $f(x) = xe^x$

$f^{(0)}(x) = f(x) = (x+0)e^x$

Dette viser at P_0 er sann.

$f^{(1)}(x) = xe^x + 1\cdot e^x = (x+1)e^x$

Dette viser at P_1 er sann.

Anta at P_n er sann for en $n \geq 1$.

Vi må vise at også P_{n+1} er sann.

$$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' \overset{P_n}{=} \left((x+n)e^x\right)'$$

$$= 1\cdot e^x + (x+n)e^x$$

$$= (x+(n+1))e^x$$

Dette viser at P_{n+1} også er sann.

b) Fra (a) følger at $f^{(k)}(0) = k$

Dermed blir Taylorrekka

$$T_n f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=1}^{n} \frac{k}{k!} x^k = \sum_{k=1}^{n} \frac{1}{(k-1)!} x^k$$

$$R_n f(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad (c \text{ mellom } 0 \text{ og } x)$$

$$= \frac{(c+n+1)e^c}{(n+1)!} x^{n+1}$$

Vi har at $(c+n+1)e^c$ er voksende i c.
For $x \in [0,1]$ så er dette derfor begrenset av verdien vi får ved å sette $c=1$

$$\left| \frac{(c+n+1)e^c}{(n+1)!} x^{n+1} \right| \leq \left| \frac{n+2}{(n+1)!} e \right|$$

Vi må ha at $\left| \frac{n+2}{(n+1)!} e \right| \leq 0.001$, eller

$$\frac{(n+1)!}{(n+2)} \geq 1000e$$

Ved å prøve oss frem ser vi at $n = 7$ er minste slike verdi.

c) (MAT-IN1105)

```
from math import *

def T(x,n):
    s = 0
    for k in range(1, n+1):
        s += x**k / factorial(k-1)
    return s
```

d) (MAT-IN1105)

```
def test_taylor():
    exact = exp(1)
    computed = T(1, 7)
    assert abs(computed - exact) <= 0.001,
        'error between exact and computed value is %s' % (computed-exact)
if __name__ == '__main__':
    test_taylor()
```

2 (MAT-INF 1100)

a) $12x_{n+2} - 7x_{n+1} + x_n = 6 \quad x_0 = 0, x_1 = \frac{2}{3}$

Karakteristiske ligning:

$$12r^2 - 7r + 1 = 0$$

$$r = \frac{7 \pm \sqrt{49-48}}{24} = \frac{7 \pm 1}{24}$$

$$r = \frac{1}{3} \text{ eller } r = \frac{1}{4}$$

Generelle løsning av homogene ligning:

$$x_n^h = C\left(\frac{1}{4}\right)^n + D\left(\frac{1}{3}\right)^n = C4^{-n} + D3^{-n}$$

Partikulær løsning: Vi prøver $x_n^p = A$:

$$12A - 7A + A = 6$$

$$6A = 6$$

$$A = 1$$

$x_n^p = 1$ part. løsning.

Generell løsning: $x_n = 1 + C4^{-n} + D3^{-n}$.

$x_0 = 0$: $\quad 1 + C + D = 0$

$x_1 = \frac{2}{3}$: $\quad 1 + \frac{C}{4} + \frac{D}{3} = \frac{2}{3}$

$$\begin{aligned} C + D &= -1 \\ 3C + 4D &= -4 \end{aligned} \quad \Bigg| \cdot(-3)$$

Løser vi disse får vi at $C = 0$, $D = -1$, slik at $\underline{\underline{x_n = 1 - 3^{-n}}}$

b) Initialbetingelsen $x_1 = \frac{2}{3}$ kan ikke representeres eksakt med 64-bits flyttall. På grunn av avrundingsfeil vil da maskinen regne ut

$$1 - (1 + \varepsilon_1) 3^{-n} + \varepsilon_2 4^{-n}$$

Den beregnede løsningen vil derfor konvergere mot 1, akkurat som den eksakte løsningen.

Oppgave 2 (MAT-IN 1105), Oppgave 3 (MAT-INF)

$$x' = \underbrace{\sin(t+x)}_{f(t,x)} \qquad x(0) = \frac{\pi}{2}$$

Her er $h = 0.1$ (skal ta et steg fra 0 til 0.1)

Euler:

$$X_1 = X_0 + hf(t_0, x_0)$$
$$= \frac{\pi}{2} + 0.1 \sin\left(0 + \frac{\pi}{2}\right) = \frac{\pi}{2} + 0.1 \approx 1.6708.$$

Euler midtpunkt

$$X_{\frac{1}{2}} = X_0 + \frac{h}{2} f(t_0, X_0)$$
$$= \frac{\pi}{2} + 0.05 \sin\left(0 + \frac{\pi}{2}\right) = \frac{\pi}{2} + 0.05$$

$$X_1 = X_0 + h f(t_{\frac{1}{2}}, X_{\frac{1}{2}})$$
$$= \frac{\pi}{2} + 0.1 \sin\left(0.05 + \frac{\pi}{2} + 0.05\right)$$
$$= \frac{\pi}{2} + 0.1 \sin\left(\frac{\pi}{2} + 0.1\right) \approx 1.6703$$

b) $x'(t) = \sin(t + x)$

$$x''(t) = \cos(t+x)(1 + x'(t))$$
$$= \cos(t+x)(1 + \sin(t+x))$$
$$= \cos(t+x) + \cos(t+x)\sin(t+x)$$
$$= \cos(t+x) + \frac{1}{2}\sin(2(t+x))$$

$$x''(0) = \cos(0+x(0)) + \frac{1}{2}\sin(2(0+x(0)))$$
$$= \cos\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\pi$$
$$= 0$$

$$x'(0) = \sin(t_0 + x_0) = \sin\left(0 + \frac{\pi}{2}\right) = 1$$

$$x(t) \approx x(0) + x'(0)t + \frac{x''(0)}{2}t^2$$
$$= \frac{\pi}{2} + t,$$

slik at tilnærmingen blir

$$\frac{\pi}{2} + 0.1 \quad \text{for } t = 0.1,$$

samme tilnærming som med Eulers metode.

Hvis vi hadde brukt ~~første~~ førsteordens Taylor med restledd:

$$x(t) = x(0) + x'(0)\,t + \frac{x''(c)}{2}\,t^2$$

Restledd:

$$R_2(t) = x''(c)\,\frac{t^2}{2} =$$

$$= \left(\cos(c+x(c)) + \frac{1}{2}\operatorname{sen}(2(c+x(c)))\right)\frac{t^2}{2}$$

Siden $|\sin x| \leq 1$ $|\cos x| \leq 1$:

$$|R_2(t)| \leq \left(1 + \frac{1}{2}\right)\frac{t^2}{2} = \frac{3}{4}t^2$$

Vi krever her at (satt inn $t=h$ over)

$$\left|\frac{3}{4}h^2\right| \leq 10^{-4},$$

$$h \leq \frac{2}{\sqrt{3}}\,10^{-2} \approx 0.0115.$$

Oppgave 4 (MAT-INF1100) Oppgave B (MAT-IN 1105)

x	0	1	3
$f(x)$	1	0	2

$$p_2(x) = c_0 + c_1 x + c_2 x(x-1)$$

$x=0$: $p_2(0) = c_0 = f(0) = \underline{1}$

$x=1$: $p_2(1) = c_0 + c_1 = f(1) = 0$

$$c_0 + c_1 = 0$$
$$1 + c_1 = 0$$
$$c_1 = \underline{-1}$$

$x=3$: $p_2(3) = c_0 + 3c_1 + c_2 \cdot 3 \cdot 2 = f(3) = 2$

$$c_0 + 3c_1 + 6c_2 = 2$$
$$1 - 3 + 6c_2 = 2$$
$$6c_2 = 2 - 1 + 3 = 4$$
$$c_2 = \frac{2}{3}$$

$$p_2(x) = c_0 + c_1 x + c_2 x(x-1)$$

$$= 1 - x + \frac{2}{3}x(x-1)$$

Tilnærming til $f'(1) \approx p_2'(1)$

$$p_2'(x) = -1 + \frac{2}{3}(x-1) + \frac{2}{3}x = \frac{4}{3}x - 1 - \frac{2}{3}$$

$$= \frac{4}{3}x - \frac{5}{3}$$

$$p_2'(1) = \frac{4}{3} - \frac{5}{3} = -\frac{1}{3}$$