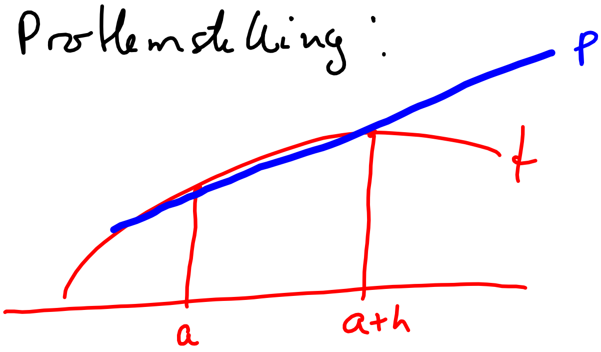
Numerisk derivasjon og integrasjon

Problemstilling:



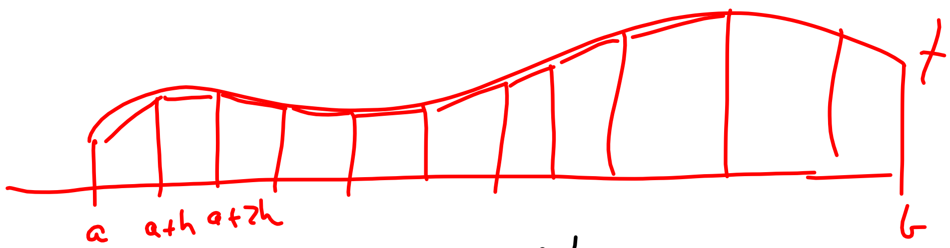
Kjenner $(a, f(a))$ og $(a+h, f(a+h))$.
Hva kan vi si om $f'(a)$ og $\int_a^{a+h} f(x)\,dx$?

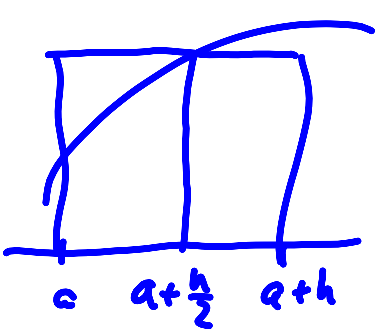
Strategi: Tilnærm f med P (her sekant).

Da er

$$f'(a) \approx P'(a) = \frac{f(a+h) - f(a)}{h},$$

$$\int_a^{a+h} f(x)\,dx \approx \int_a^{a+h} P(x)\,dx = \frac{f(a) + f(a+h)}{2} \cdot h$$





$$\int_a^{a+h} f(x)\,dx \approx h \cdot f\left(a + \frac{h}{2}\right)$$

$$f(x) \approx P(x) = f\left(a + \frac{h}{2}\right)$$

Eksempel, Taylor med restledd.

Vi ser på funksjonen $f(x) = \ln(1+x)$

Når konvergerer Taylorpolynomet for positive x?

$$T_n f(x) = f(a) + (x-a) f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

$a = 0$, Må finne $f^{(k)}(0)$, $k = 0, 1, \ldots, n$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}, \quad f''(x) = -(1+x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3}, \quad f''''(x) = -6(1+x)^{-4}$$

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}, \quad n \geq 1$$

$$T_n f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0)$$

$$= 0 + x \cdot 1 + \frac{x^2}{2} \cdot (-1) + \frac{x^3}{6} \cdot 2 + \frac{x^4}{4!} \cdot (-6)$$

$$+ \cdots + \frac{x^n}{n!} \cdot (-1)^{n+1} (n-1)!$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n}$$

Feilledd

$$R_n f(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \quad \xi \in (0,x)$$

$$= \frac{x^{n+1}}{(n+1)!} (-1)^{n+2} n! (1+\xi)^{-n-1} \qquad (1+\xi)^{-(n+1)}$$

$$= (-1)^{n+2} \frac{x^{n+1}}{n+1} \frac{1}{(1+\xi)^{n+1}}$$

$$|R_n f(x)| = \frac{x^{n+1}}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}} \overset{\xi=0}{\leq} \frac{x^{n+1}}{n+1} .$$

Konv når $x < 1$. og $x = 1$