Numerisk Integrasjon

Gitt en reell funksjon $f : [a,b] \to \mathbb{R}$, så ønsker vi å approksimere integralet

$$I[f] = \int_a^b f(x)\,dx,$$

ved å bruke en kvadraturregel,

$$I_n[f] = \sum_{i=1}^{n} w_i f(x_i),$$

hvor $a \leq x_1 < x_2 < \dots < x_n \leq b$ og $w_1, w_2, \dots, w_n$ er reelle tall, "vekter".

Mest vanlig måte å velge vektene er å bruke polynominterpolasjon m.h.t. punktene $x_1, \dots, x_n$.

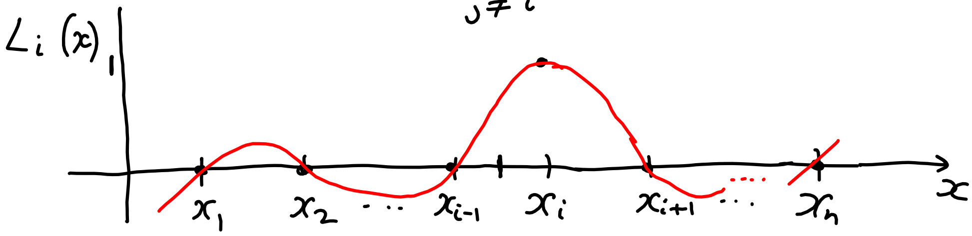
La $p(x)$ være polynomet av grad $\leq n-1$ som interpolerer f i $x_1, x_2, .., x_n$.

Det hjelper å bruke Lagrangeformen av p, d.v.s.,

$$p(x) = \sum_{i=1}^{n} L_i(x) f(x_i), \qquad (*)$$

hvor

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right).$$



Ser at $L_i(x_i) = 1$, $L_i(x_k) = 0$, $k \neq i$

Nå definér $I_n[f] = I[p]$

$$= \int_a^b p(x)\,dx = \int_a^b \sum_{i=1}^{n} L_i(x) f(x_i)\,dx$$

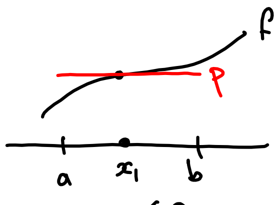
$$= \sum_{i=1}^{n} \left(\int_a^b L_i(x)\,dx \right) f(x_i)$$

$$= \sum_{i=1}^{n} w_i f(x_i),$$

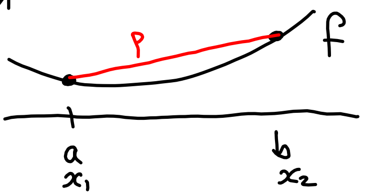
hvor $w_i = \int_a^b L_i(x)\,dx$.

<u>Eksempler</u> n antall punkt:

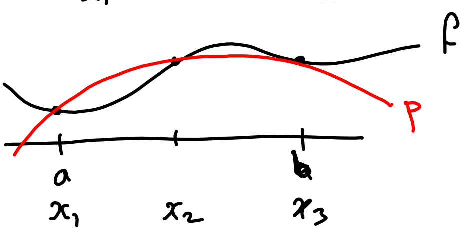
$n=1$: midtpunktregelen $\longrightarrow$



$n=2$: trapesregelen

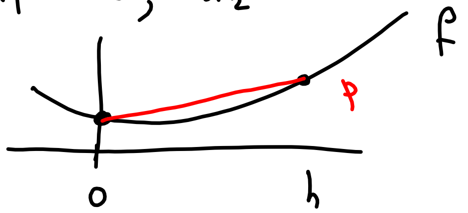


$n=3$: Simpsons regel



Trapesregelen, $n = 2$. Anta at $[a,b] = [0,h]$.

La $x_1 = 0$, $x_2 = h$



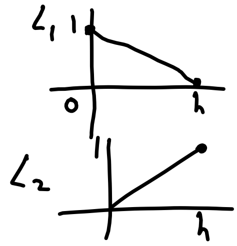
$$p(x) = \sum_{i=1}^{2} L_i(x) f(x_i) = L_1(x) f(x_1) + L_2(x) f(x_2),$$

$$L_1(x) = \frac{h-x}{h}, \quad L_2(x) = \frac{x}{h}.$$

(sjekk: $L_1(0) = 1$, $L_1(h) = 0$, $L_2(0) = 0$, $L_2(h) = 1$)

$$w_1 = \int_0^h L_1(x)\,dx = \int_0^h \frac{h-x}{h}\,dx = \frac{h}{2}$$

$$w_2 = \int_0^h L_2(x)\,dx = \int_0^h \frac{x}{h}\,dx = \frac{h}{2}$$



$$\Rightarrow \boxed{I_2[f] = \frac{h}{2}\left(f(0) + f(h)\right)}$$ "trapesregelen"

Husk midtpunktregelen: $\boxed{I_1[f] = h\, f\left(\frac{h}{2}\right)}$.

Simpsonsregel $n=3$ (3 punkt) La $[a,b]=[-h,h]$

La $x_1=-h$, $x_2=0$, $x_3=h$.

$$L_i(x)=\prod_{\substack{j=1\\ j\neq i}}^{3}\frac{x-x_j}{x_i-x_j}\ \cdot (*) \Rightarrow$$

$$L_1(x)=\frac{x(x-h)}{2h^2},\quad L_2(x)=\frac{h^2-x^2}{h^2},\quad L_3(x)=\frac{x(x+h)}{2h^2}.$$

$$w_1=\frac{h}{3},\quad w_2=\frac{4h}{3},\quad w_3=\frac{h}{3}.$$

$$\Rightarrow \boxed{I_3[f]=\frac{h}{3}\left(f(-h)+4f(0)+f(h)\right)}$$

Simpsons regel.

Detaljer:

$$L_1(x)=\prod_{j=2}^{3}\left(\frac{x-x_j}{x_1-x_j}\right)=\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)$$

$x_1=-h$, $x_2=0$, $x_3=h$

$$\Rightarrow L_1(x)=\left(\frac{x-0}{-h-0}\right)\left(\frac{x-h}{-h-h}\right)=\frac{x(x-h)}{2h^2}$$

$$w_1=\int_{-h}^{h}L_1(x)\,dx=\int_{-h}^{h}\frac{x(x-h)}{2h^2}\,dx$$

$$=\frac{1}{2h^2}\int_{-h}^{h}(x^2-hx)\,dx=\frac{1}{2h^2}\left[\frac{x^3}{3}-\frac{hx^2}{2}\right]_{-h}^{h}=\frac{h}{3}.$$

Sammensatte regler

Interval $[a,b]$. Ønsker å finne $\int_a^b f(x)\,dx$.

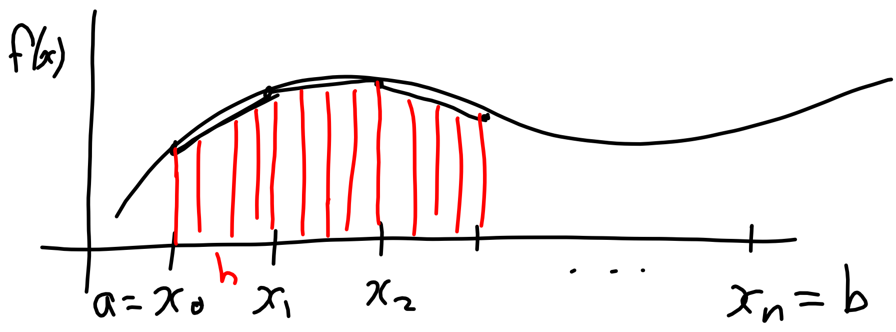
Dele opp $[a,b]$ i biter: La $x_i = a + ih$, $i = 0, 1, \dots, n$ hvor $h = \frac{b-a}{n}$.

Da er $\int_a^b f(x)\,\cancel{d}x = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$

Bruk: en kvadraturregel på hvert integral $\int_{x_i}^{x_{i+1}} f(x)\,dx$.

Eksempel Sammensatte trapesregelen:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right)$$
$$= \frac{h}{2} f(x_0) + h \sum_{i=1}^{n-1} f(x_i) + \frac{h}{2} f(x_n)$$



Sammensatte Simpsons regel La $n = 2m$.

$$\int_a^b f(x)\,dx = \sum_{i=0}^{m-1} \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx$$

$$= \sum_{i=0}^{m-1} \frac{h}{3}\left(f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right)$$

$$= \frac{h}{3}\left(f(x_0) + 4S_1 + 2S_2 + f(x_n)\right),$$

hvor $S_1 = f(x_1) + f(x_3) + \dots + f(x_{n-1})$

$S_2 = f(x_2) + f(x_4) + \dots + f(x_{n-2})$.

Feilen ? Har sagt at $I_n[f] = I[p]$

hvor $p(x)$ interpolerer f i $x_1, x_2, \dots, x_n$.

p har grad $\leq n-1$.

Det betyr at feilen er

$$I[f] - I_n[f] = \int_a^b f(x)\,dx - \int_a^b p(x)\,dx$$

$$= \int_a^b \big(f(x) - p(x)\big)\,dx .$$

Så hva er $f(x) - p(x)$?

Svaret : $\boxed{f(x) - p(x) = (x-x_1)(x-x_2)\dots(x-x_n)[x_1, x_2, \dots, x_n, x]f}$

hvor $[x_1, x_2, \dots, x_n, x]f$ er den dividerte differanse av f i punktene $x_1, \dots, x_n, x$.

Dividerte differanser :

$$[x_0]f = f(x_0)$$

$$[x_0, x_1]f = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$[x_0, x_1, x_2]f = \frac{[x_1, x_2]f - [x_0, x_1]f}{x_2 - x_0}$$

$$\vdots$$

$$[x_0, x_1, \dots, x_k]f = \frac{[x_1, \dots, x_k]f - [x_0, \dots, x_{k-1}]f}{x_k - x_0}.$$

Kan vise at det finnes ξ s.a. $[x_0, \dots, x_k]f = \dfrac{f^{(k)}(\xi)}{k!}$,

Da er : $I[f] - I_n[f] = \int_a^b (x-x_1)\dots(x-x_n)[x_1, \dots, x_n, x]f\,dx$

$$= \int_a^b (x-x_1)\dots(x-x_n)\frac{f^{(n)}(\xi_x)}{n!}\,dx$$

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