Induksjon, 1.2 Binomialteoremet, 1.4

Induksjonsprinsippet. Har et utsagn P_n, avhenger av n, $n \in \mathbb{N}$. Anta at

(i) P_1 er sant

(ii) Dersom P_k er sant, er også P_{k+1} sant

Da er P_n sant for alle $n \in \mathbb{N}$.

Eksempel Bernoullis ulikhet

P_n : $(1+x)^n \geq 1+nx$ for alle $x \geq -1$, $n \in \mathbb{N}$.

Induksjon:

(i) P_1 ? $(1+x)^1 \geq 1+x$ sant ✓

(ii) Anta at P_k er sant, d.v.s.

anta at $(1+x)^k \geq 1+kx$ for $x \geq -1$ (*)

Vi må vise at P_{k+1} er sant, d.v.s., at

$$(1+x)^{k+1} \geq \underline{1+(k+1)x} \quad \text{for } x \geq -1. \ (*)$$

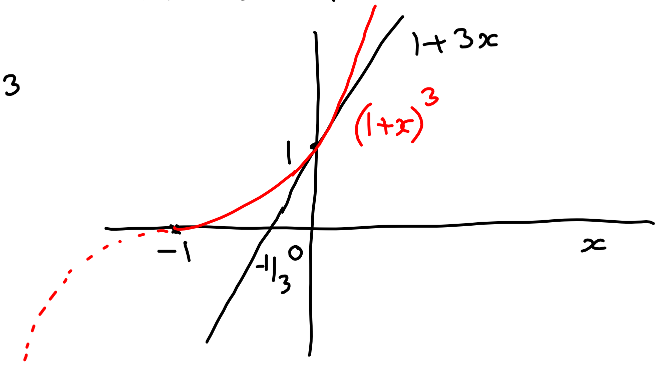
$$\text{V.S. } (1+x)^{k+1} = \underbrace{(1+x)}_{\geq 0}(1+x)^k \geq (1+x)(1+kx) \quad (*)$$

$$= 1 + (k+1)x + \underbrace{kx^2}_{\geq 0}$$

$$\geq 1+(k+1)x$$

Har vist at P_{k+1} er sant. Beviset er ferdig $\square$

$n = 3$



$$4 \geq 3$$

$$\Rightarrow \quad 2 \cdot 4 \geq 2 \cdot 3 \qquad (-2)\cdot 4 \leq (-2)\cdot 3$$

$$8 \geq 6 \qquad -8 \leq -6$$

$a \geq b$ $\qquad$ $b \geq c$

$4 \geq 2$ og $2 \geq 1$

$$\Rightarrow \quad 4 \geq 1$$

$$a \geq c$$

Variasjoner av induksjonsprinsippet

La n_0 være et heltall.

La P_n et utsagn for alle $n \geq n_0$

Anta at

(i) P_{n_0} er sant

(ii) Dersom P_m er sant for alle m slik at $n_0 \leq m < k$, er også P_k sant.

Da er P_n sant, alle $n \geq n_0$.

Binomial teoremet

Hvordan regner vi ut $(a+b)^n$.

$n \in \mathbb{N}$, $a, b \in \mathbb{R}$.

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Generelt. Hva er $(a+b)^n$?

$$(a+b)^n = a^n + c_1 a^{n-1} b^1 + c_2 a^{n-2} b^2 + \dots + c_{n-1} a b^{n-1} + b^n.$$

Hva er koeffisientene $c_1, c_2, \dots, c_{n-1}$?

Pascal's trekant

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Binomial koeffisienter

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!} \quad , \quad n \in \mathbb{N}, \quad 0 \le i \le n.$$

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

$$2! = 1 \cdot 2 = 2$$

$$3! = 1 \cdot 2 \cdot 3 = 6$$

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Eksempel

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{2 \cdot 1 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 10$$

Pascal's trekant

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0}\,1 & & & \\
 & & \binom{1}{0}\,1 & & \binom{1}{1}\,1 & & \\
 & \binom{2}{0}\,1 & & \binom{2}{1}\,2 & & \binom{2}{2}\,1 & \\
\binom{3}{0}\,1 & & \binom{3}{1}\,3 & & \binom{3}{2}\,3 & & \binom{3}{3}\,1
\end{array}
$$

$$\binom{3}{2} = \frac{3!}{2!\,1!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3$$

$$\binom{3}{1} = \frac{3!}{1!\,2!} = \binom{3}{2} = 3$$

Merk: symmetri : $\binom{n}{i} = \binom{n}{n-i}$

Fordi $\binom{n}{n-i} = \frac{n!}{(n-i)!\,(n-(n-i))!} = \frac{n!}{(n-i)!\,i!} = \frac{n!}{i!\,(n-i)!} = \binom{n}{i}$

Lemma 1.4.4

For alle $n \in \mathbb{N}$ og i med $0 \leq i \leq n$,

er
$$\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}.$$

Bevis

$$\binom{n}{i-1} + \binom{n}{i} = \frac{n!}{(i-1)!\,(n-(i-1))!} + \frac{n!}{i!\,(n-i)!}$$

$$= \frac{n!}{(i-1)!\,(n-i+1)!} + \frac{n!}{i!\,(n-i)!}$$

$n!$ er en felles faktor.

Fellesnevner : $i!\,(n-i+1)!$

$$= \frac{n!}{i!\,(n-i+1)!}\Big(i + (n-i+1)\Big)$$

$$= \frac{n!}{i!\,(n-i+1)!}(n+1)$$

$$= \frac{(n+1)!}{i!\,(n-i+1)!} = \binom{n+1}{i}$$

$\square$

Binomialteoremet

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \qquad (*)$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots$$

$$= a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots$$

$$+ b^n.$$

Hvorfor? $\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{1 \; n!} = 1$ fordi $0! = 1$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 1}{1 \cdot (n-1) \cdot (n-2) \cdots 1} = n$$

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n\,(n-1)}{2}$$

<u>Bevis</u> Induksjon på n.

P_n : $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$.

(i) P_1 ? V.S. $(a+b)^1$. H.S. $\sum_{i=0}^{1} \binom{1}{i} a^{1-i} b^i = \binom{1}{0}a + \binom{1}{1} b = a+b$.

P_1 er sant.

(ii) Anta P_k er sant. d.v.s. $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$

Må vise at $(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$

$$\text{V.S.} = (a+b)(a+b)^k \overset{P_k}{=} (a+b) \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1}$$

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^{i+1} + b^{k+1}$$

$$= \text{"} + \text{"} + \sum_{i=1}^{k} \binom{k}{i-1} a^{k-(i-1)} b^{(i-1)+1} + \text{"}$$

$$\text{"} + \text{"} + \sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^i + \text{"}$$

$$= a^{k+1} + \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1}\right) a^{k+1-i} b^i + b^{k+1}$$

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} a^{k+1-i} b^i + b^{k+1}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i .$$

$\square$

Eksempel :

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}$$

$\Rightarrow$ summen av $\binom{n}{i}$ (over i) $= 2^n$

Eks.

$1 + 1 = 2$

$1 + 2 + 1 = 4$

$1 + 3 + 3 + 1 = 8$