Uke 7 Taylorapproksimasjon, polynominterpolasjon.

Velkommen: MAT-IN1105:

Grupper 5 og 7 gruppetime 1215-1400 mand.

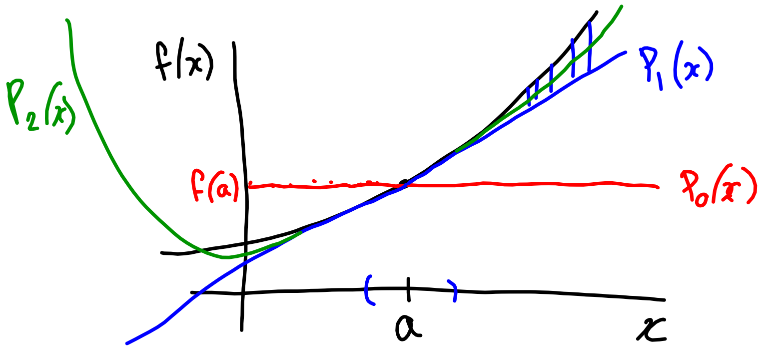
5: VE26, 7: 108 NHA

Idag Taylorapproksimasjon.

Gitt en funksjon $f(x)$, ønsker å approksimere f med et polynom $P_n(x)$ med grad $\leq n$.

Anta at f er glatt nok i et punkt a.

Vi velger P_n slik at $P_n(a) = f(a)$, $P_n'(a) = f'(a)$, og $P_n''(a) = f''(a)$, ..., $P_n^{(n)}(a) = f^{(n)}(a)$.



$\underline{n=0}$ P_0 må være en konstant, $P_0(x) = f(a)$.

$\underline{n=1}$ $P_1(x) = b_0 + b_1 x$, ønsker $P_1(a) = f(a)$, $P_1'(a) = f'(a)$.

Bedre å skrive P_1 på formen

$$P_1(x) = c_0 + c_1(x-a).$$

$$\Rightarrow \quad P_1'(x) = c_1$$

$$\Rightarrow \quad \begin{aligned} P_1(a) &= c_0 \\ P_1'(a) &= c_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \end{aligned}$$

$$\Rightarrow \quad \boxed{P_1(x) = f(a) + f'(a)(x-a)}.$$

$\underline{n=2}$. Ønsker $P_2(x)$, slik at $P_2(a)=f(a)$, $P_2'(a)=f'(a)$, $P_2''(a)=f''(a)$

Hjelper å skrive $P_2(x)$ som $P_2(x)=c_0+c_1(x-a)+c_2(x-a)^2$.

$$\Rightarrow \quad P_2'(x) = c_1 + 2c_2(x-a)$$

$$P_2''(x) = 2c_2$$

$$x=a \quad \Rightarrow \quad \left.\begin{aligned} P_2(a) &= c_0 \\ P_2'(a) &= c_1 \\ P_2''(a) &= 2c_2. \end{aligned}\right\} \Rightarrow \quad \begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{1}{2} f''(a) \end{aligned}$$

$$\Rightarrow \quad P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2.$$

Generell n ? Ønsker $P_n(x)$, poly. grad $\leq n$, slik at

$$P_n^{(k)}(a) = f^{(k)}(a), \quad k = 0, 1, \ldots, n.$$

La
$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n.$$
$$= \sum_{i=0}^{n} c_i (x-a)^i$$

$$P_n'(x) = \sum_{i=0}^{n} i c_i (x-a)^{i-1} = \sum_{i=1}^{n} i c_i (x-a)^{i-1}$$

$$P_n''(x) = \sum_{i=2}^{n} i(i-1) c_i (x-a)^{i-2}$$

.....
$$P_n^{(k)}(x) = \sum_{i=k}^{n} i(i-1)\cdots(i-k+1)\, c_i (x-a)^{i-k}$$
$$= \sum_{i=k}^{n} \frac{i!}{(i-k)!} c_i (x-a)^{i-k}$$

La $x = a$:
$$P_n^{(k)}(a) = k!\, c_k$$

$$\Rightarrow \quad c_k = \frac{1}{k!} P_n^{(k)}(a) = \frac{1}{k!} f^{(k)}(a)$$

$$\Rightarrow \quad \boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k}.$$

Skriver $P_n(x) = T_n f(x)$. Dette er Taylor approksimasjon til f av orden n.

(merk: graden til $T_n f(x)$ kan være mindre enn n).

Eksempel. $f(x) = e^x = \exp(x)$

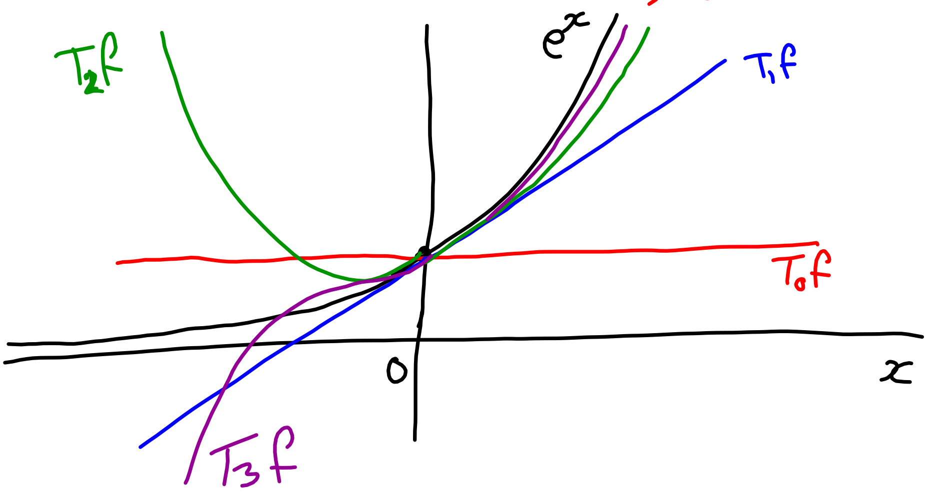
Finn Taylor app. til f av orden n, om punktet 0.

$f(x) = e^x$	$f(0) = 1$
$f'(x) = e^x$	$f'(0) = 1$
$f''(x) = e^x$	$\vdots$
$\vdots$	
$f^{(n)}(x) = e^x$	$f^{(n)}(0) = 1$

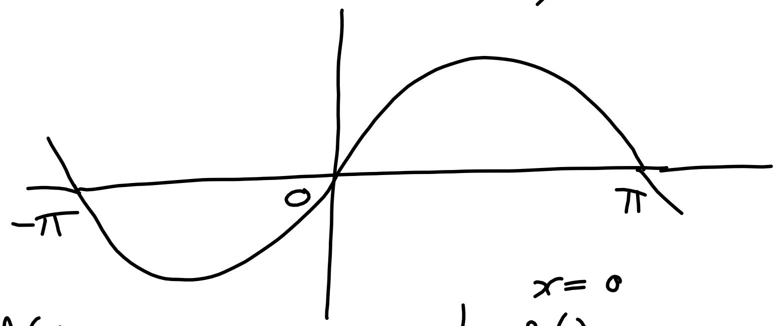
$$\Rightarrow \quad T_n e^x = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}. \quad (*)$$

Husk: $k! = 1.2.3\ldots k$, $1! = 1$, $0! = 1$.



Eksempel $f(x) = \sin x$, la $a = 0$.



	$x = 0$
$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x \quad \ldots$	$f^{(5)}(0) = 1$

$$\Rightarrow T_n \sin x = \frac{0}{0!}(x-0)^0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 + \frac{(-1)}{3!}(x-0)^3$$

$$+ \frac{0}{4!}(x-0)^4 + \frac{1}{5!}(x-0)^5 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Anta $n = 2k-1$. Da er

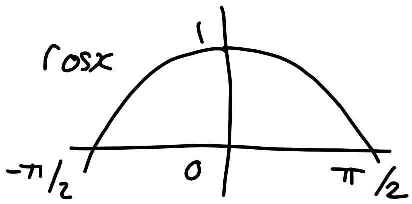
$$T_{2k-1} \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}$$

n partall! $n = 2k$

$$T_{2k} \sin x = T_{2k-1} \sin x$$

Graden til $T_{2k} \sin x$ er bare $2k-1$.

$T_n \cos x$?



$$T_n \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

La $n \to \infty$. Hva skjer?

Da får vi

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (*)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Euler setning? $e^{ix} = \cos x + i \sin x$, $i = \sqrt{-1}$

$$(*) \Rightarrow e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos x + i \sin x$$

Man kan også "vise":

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$