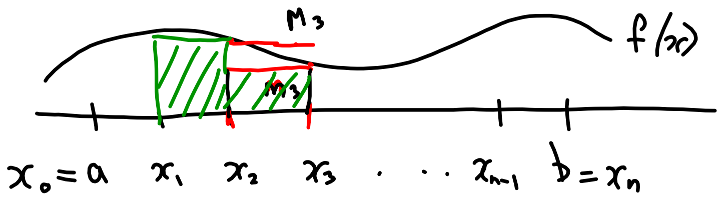
Numerisk Integrasjon   Kap. 12, Kompendiet.

Ønsker å finne et integral : $\int_a^b f(x)\,dx$.

Hva er definisjonen av $\int_a^b f(x)\,dx$?

Vi deler intervallet $[a,b]$ i n biter :



La $x = a$
$b = x_n$

La $m_i = \min\limits_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \max\limits_{x \in [x_{i-1}, x_i]} f(x)$,

$i = 1, 2, \ldots, n$.

La $\underline{I}_n = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$, $\overline{I}_n = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$

Når $n \to \infty$. Grensene :

$$\underline{I} = \sup \underline{I}_n , \quad \overline{I} = \inf \overline{I}_n$$

Hvis $\underline{I} = \overline{I}$ da er f er integrerbar

og $\int_a^b f(x)\,dx = \underline{I} = \overline{I}$.

Numerisk metode for å beregne $\int_a^b f(x)\,dx$.

En enkel metode: <u>midtpunktmetoden</u>:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1/2})(x_i - x_{i-1}),$$

hvor $x_{i-1/2} = \dfrac{x_{i-1}+x_i}{2}$

Vi antar igjen at vi deler $[a,b]$ uniformt, d.v.s. når vi velger n, vi lar $h = \dfrac{b-a}{n}$, og vi setter $x_i = a + ih$, $i = 0, 1, \ldots, n$.

Da er $x_i - x_{i-1} = h$ for alle i, og midtpunktmetoden er:

$$\int_a^b f(x)\,dx \approx I(n) = h\sum_{i=1}^{n} f(x_{i-1/2}).$$

Regn ut $I(1)$, $I(2)$, $I(4)$, $I(8)$, Hver gang fordobler n, eller tilsvarende, halverer h. Fortsetter inntil forskjellen mellom $I(2^k)$ og $I(2^{k-1})$ er "liten", for eksempel når

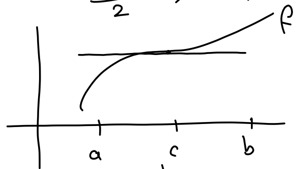
$$|I(2^k) - I(2^{k-1})| < 10^{-12}.$$

Feilanalyse. To steg: første lokal feilanalyse, etterpå den globale feilen.

Lokal feil. Anta $n=1$. Da er tilnærmingen

$$\int_a^b f(x)\,dx \approx (b-a)\,f(c),$$

hvor $c = \frac{a+b}{2}$, midtpunktet av $[a,b]$.

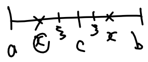


Feilen er $E = \int_a^b f(x)\,dx - (b-a)f(c)$.

Kan bruke Taylor approksimasjon,

$$f(x) = \underbrace{f(c) + (x-c)f'(c)}_{T_1 f(x)} + \underbrace{\frac{(x-c)^2}{2!} f''(\xi_x)}_{R_1 f(x)}$$

hvor ξ er et punkt mellom x og c



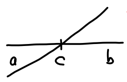
. Merk at $\xi_x \in [a,b]$.

Nå integrer ligningen:

$$\int_a^b f(x)\,dx = \int_a^b f(c)\,dx + \int_a^b (x-c)f'(c)\,dx + \int_a^b \frac{(x-c)^2}{2!} f''(\xi_x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = f(c)(b-a) + f'(c)\int_a^b (x-c)\,dx + \frac{1}{2}\int_a^b (x-c)^2 f''(\xi_x)\,dx$$

$$\int_a^b (x-c)\,dx = 0.$$



$b-c = \frac{1}{2}(b-a)$

$c-a = \frac{1}{2}(b-a)$

Hvorfor? $$\int_a^b (x-c)\,dx = \left[\frac{(x-c)^2}{2}\right]_{x=a}^{x=b} = \frac{(b-c)^2}{2} - \frac{(c-a)^2}{2} = \frac{(b-a)^2}{8} - \frac{(b-a)^2}{8} = 0.$$

Derfor er feilen

$$E = \left|\int_a^b f(x) - (b-a)f(c)\right| = \frac{1}{2}\left|\int_a^b (x-c)^2 f''(\xi_x)\,dx\right|$$

$$\leq \frac{1}{2}\int_a^b \left|(x-c)^2 f''(\xi_x)\right| dx$$

$$= \frac{1}{2}\int_a^b |x-c|^2\, |f''(\xi_x)|\,dx$$

$$\leq \frac{1}{2}\int_a^b |x-c|^2\, M\,dx,$$

hvor $M = \max_{a \leq x \leq b} |f''(x)|$

$$= \frac{1}{2} M \int_a^b (x-c)^2\,dx$$

$$= \frac{1}{2} M \left[\frac{(x-c)^3}{3}\right]_{x=a}^{x=b}$$

$$= \frac{1}{6} M \left((b-c)^3 - (a-c)^3\right) \quad {}^{+\,(c-a)^3}$$

$b-c = \frac{1}{2}(b-a)$

$-(a-c) = c-a = \frac{1}{2}(b-a)$

$$= \frac{1}{6} M \left(\frac{(b-a)^3}{8} + \frac{(b-a)^3}{8}\right)$$

$$= \frac{M}{24}(b-a)^3.$$

Global feil : Velg n, $h = \frac{b-a}{n}$, $x_i = a + ih$,

$i = 0, 1, \ldots, n.$

$$E_g = \int_a^b f(x)\,dx - \underbrace{h \sum_{i=1}^{n} f(x_{i-1/2})}_{\text{midtpunktmetoden}}$$

$$x_{i-1/2} = \frac{x_{i-1} + x_i}{2}.$$

Ser at $E_g = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx - h \sum_{i=1}^{n} f(x_{i-1/2})$

$$= \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - h f(x_{i-1/2}) \right)$$

$$\Rightarrow |E_g| \leq \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - h f(x_{i-1/2}) \right|$$

$$\leq \sum_{i=1}^{n} \frac{M_i h^3}{24},$$

hvor $M_i = \max\limits_{x_{i-1} \leq x \leq x_i} |f''(x)|.$

Men $M_i \leq M$, hvor $M = \max\limits_{a \leq x \leq b} |f''(x)|$

Derfor er $|E_g| \leq M \sum_{i=1}^{n} \frac{h^3}{24}$

$$= \frac{M h^3}{24} \sum_{i=1}^{n} 1$$

$$= \frac{M h^3}{24} n$$

Husk at $h = \frac{b-a}{n}$. $\Rightarrow nh = b-a$

$$\Rightarrow \boxed{|E_g| \leq \frac{M h^2 (b-a)}{24}}.$$

Ser at når vi halverer h, så går feilen ned med faktor 4.