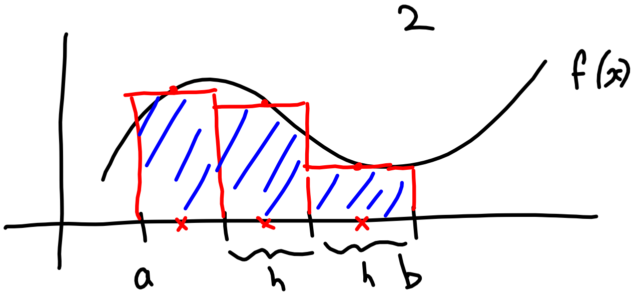
Numerisk Integrasjon.

Husk midtpunktregelen: ønsker å approksimere $\int_a^b f(x)\,dx$. Velger n, og deler $[a,b]$ i biter:

La $h = \frac{b-a}{n}$, og la $x_i = a + ih$, $i = 0, 1, \ldots, n$:

Regelen er $\int_a^b f(x)\,dx \approx h \sum_{i=1}^{n} f(x_{i-1/2}) = \sum_{i=1}^{n} \underbrace{h f(x_{i-1/2})}_{\text{areal i rektangel.}}$

hvor $x_{i-1/2} = \frac{x_{i-1} + x_i}{2}$.

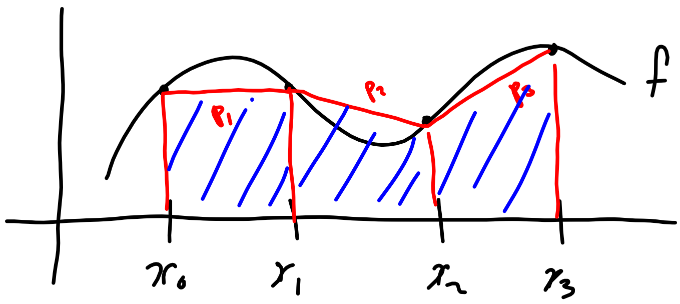


$O(h^2)$.

<u>Alternative metoder:</u> I hvert intervall $[x_{i-1}, x_i]$ approksimer f med et polynom p_i og da er

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} p_i(x)\,dx \quad (*)$$

<u>Eksempel</u> p_i polynomet grad 1 som interpolerer f i x_{i-1} og x_i: $p_i(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i)$.



Lett å vise at

$$\int_{x_{i-1}}^{x_i} p_i(x)\,dx = \frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

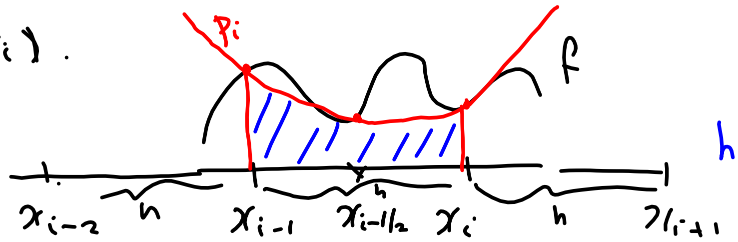
Da er

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

$$= \frac{h}{2} f(x_0) + h \sum_{i=1}^{n-1} f(x_i) + \frac{h}{2} f(x_n).$$

Dette kalles trapesregelen. $O(h^2)$.

Eksempel 2. Vi lar p_i være polynomet av grad ≤ 2 slik at $p_i(x_{i-1}) = f(x_{i-1})$, $p(x_{i-1/2}) = f(x_{i-1/2})$, $p_i(x_i) = f(x_i)$.



$h = x_i - x_{i-1}$

Kan vise at $\displaystyle\int_{x_{i-1}}^{x_i} p_i(x)\,dx = \frac{h}{6}\left(f(x_{i-1}) + 4f(x_{i-1/2}) + f(x_i)\right)$.

Da er
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{h}{6}\left(f(x_{i-1}) + 4f(x_{i-1/2}) + f(x_i)\right)$$
$$= \frac{h}{6}\left(f(x_0) + 4S_1 + 2S_2 + f(x_n)\right),$$

hvor
$$S_1 = f(x_{1/2}) + f(x_{3/2}) + \dots + f(x_{n-1/2})$$
$$S_2 = f(x_1) + f(x_2) + \dots + f(x_{n-1}).$$

Simpsonsregel. Nøyaktigheten er $O(h^4)$.

For generell grad av p_i får vi de så kalte Newton–Cotes regeler.

Detaljer. Trapesregelen:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} p_i(x)\,dx,$$

$$p_i(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i)$$

Hvorfor? $p_i(x_{i-1}) = 1.f(x_{i-1}) + 0.f(x_i) = f(x_{i-1})$

$p_i(x_i) = 0.f(x_{i-1}) + 1.f(x_i) = f(x_i).$

Husk $h = x_i - x_{i-1}$

$$\text{Finn } \int_{x_{i-1}}^{x_i} p_i(x)\,dx = \left(\int_{x_{i-1}}^{x_i} \frac{x_i - x}{h}\,dx\right) f(x_{i-1}) + \left(\int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h}\,dx\right) f(x_i)$$

$$= \left[-\frac{(x_i - x)^2}{2h}\right]_{x_{i-1}}^{x_i} f(x_{i-1}) + \left[\frac{(x - x_{i-1})^2}{2h}\right]_{x_{i-1}}^{x_i} f(x_i)$$

$$-(-1)\frac{h^2}{2h} f(x_{i-1}) + \frac{h^2}{2h} f(x_i)$$

$$= \frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right).$$

Hva er feilen? Kan vise at

$$f(x) - p_i(x) = (x - x_{i-1})(x - x_i)\frac{f''(\xi_i)}{2!} \quad (*)$$

hvor $\xi_i \in (x_{i-1}, x_i)$.

Integrere:

$$\int_{x_{i-1}}^{x_i} f(x)\,dx - \int_{x_{i-1}}^{x_i} p_i(x)\,dx = \frac{1}{2}\int_{x_{i-1}}^{x_i} \underbrace{(x - x_{i-1})(x - x_i)}_{\leq 0} f''(\xi_i)\,dx$$

$$= \frac{f''(c_i)}{2}\int_{x_{i-1}}^{x_i} (x - x_{i-1})(x - x_i)\,dx$$

$$= \frac{f''(c_i)}{2}\int_0^h y(y - h)\,dy$$

(la $y = x - x_{i-1}$, $dy = dx$)

$$= \frac{f''(c_i)}{2}\int_0^h y^2 - hy\,dy$$

$$= \frac{f''(c_i)}{2}\left[\frac{y^3}{3} - \frac{hy^2}{2}\right]_0^h$$

$$= \frac{f''(c_i)}{2}\left[\frac{h^3}{3} - \frac{h.h^2}{2}\right]$$

$$= -\frac{f''(c_i)\,h^3}{12}. \quad (**)$$

Lokale feilen.

Globale feilen er:

$$\int_a^b f(x)\,dx - \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} p_i(x)\,dx$$

$$= \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left(f(x) - p_i(x)\right)dx$$

$$= \sum_{i=1}^{n}\left(-\frac{f''(c_i)}{12}h^3\right)$$

$$= -\frac{1}{12}h^3\sum_{i=1}^{n} f''(c_i)$$

$$= -\frac{1}{12}h^3\,n\,f''(c), \quad c \in (a, b)$$

$$= -\frac{1}{12}(b-a)h^2 f''(c). \quad = O(h^2).$$