Oblig 2:

Oppgave 1. Finn Taylorapproks. av orden 3 til $f(x) = \arctan(x)$ i punktet $a = 0$.

Svar: $f'(x) = \dfrac{1}{1+x^2}$, $f''(x) = \dfrac{-2x}{(1+x^2)^2}$

$$f'''(x) = \frac{(1+x^2)^{\cancel{2}\,1}\cdot(-2) - 2\cancel{(1+x^2)}\cdot 2x\cdot(-2x)}{(1+x^2)^{\cancel{4}\,3}} = \frac{-2+6x^2}{(1+x^2)^3}.$$

$$\Rightarrow f(0) = 0,\quad f'(0) = 1,\quad f''(0) = 0,\quad f'''(0) = -2.$$

$$T_3 f(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$= \cancel{f(0)}^{\,0} + f'(0)x + \cancel{\frac{f''(0)}{2!}}^{\,0} x^2 + \frac{f'''(0)x^3}{3!}$$

$$= x - \frac{2x^3}{2\cdot 3} = x - \frac{x^3}{3}.$$

1(b) lag et plot av $f(x) = \ln(x)$, og T. app. av orden 1, 2, 3. for $x \in [0.05, 1.95]$. rundt $a = 1$.

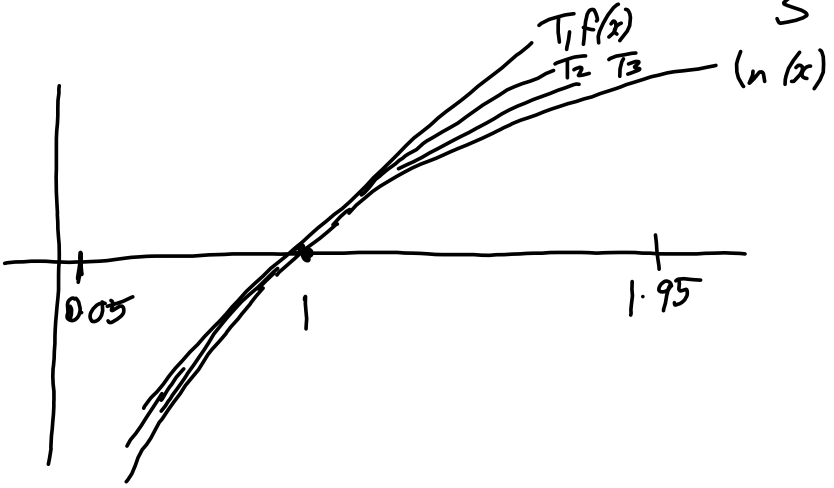
$f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$.

$f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$.

$$T_3 f(x) = \sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2 \cdot (x-1)^3}{3!}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$



Oppgave 2. (a). Gitt funksjonen $f(x) = \frac{1}{x^2}$

Vis ved induksjon at $f^{(k)}(x) = \frac{(-1)^k (k+1)!}{x^{k+2}}$, $k \geq 0$.

Svar. La P_k være utsagnet at $f^{(k)}(x) = \frac{(-1)^k (k+1)!}{x^{k+2}}$ (*)

(i) Vis at P_0 er sant. P_0 er: $f(x) = \frac{(-1)^0 (0+1)}{x^{0+2}} = \frac{1}{x^2}$.
Stemmer. P_0 er sant. ✓

(ii) Anta at P_k er sant for en vilkårlig $k \geq 0$.
Vis at P_{k+1} er sant.

P_{k+1}? $f^{(k+1)}(x) = \frac{d}{dx} f^{(k)}(x) = \frac{d}{dk}\left(\frac{(-1)^k (k+1)!}{x^{k+2}}\right)$

$$= (-1)^k (k+1)! \frac{d}{dx}\left(\frac{1}{x^{k+2}}\right)$$

$$= (-1)^k (k+1)! \frac{d}{dx} x^{-(k+2)}$$

$$= (-1)^k (k+1)! (-(k+2))\, x^{-(k+2)-1}$$

$$= (-1)^{k+1} (k+2)! \frac{1}{x^{k+3}}.$$

Det viser at P_{k+1} er sant. ✓

b) Finn Tay. App. til f om punktet $a = 1$
og et uttrykk for restleddet $R_n f(x)$.

$$T_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} \frac{(-1)^k (k+1)!}{k!}(x-1)^k$$

Men $(k+1)! = k!\,(k+1)$ $\Rightarrow$ $T_n f(x) = \sum_{k=0}^{n} (-1)^k (k+1)(x-1)^k$.

$$\boxed{R_n f(x) = \frac{f^{(n+1)}(c)\,(x-1)^{n+1}}{(n+1)!}}$$, hvor c er mellom x og 1.

$$= \frac{(-1)^{n+1}(n+2)}{c^{n+3}}(x-1)^{n+1}.$$

2(c) Finn N slik at for alle $n \geq N$ og for alle $x \in [1, 1.25]$ er feilen i $T_n f(x)$ mindre enn 0.02.

<u>Svar</u> Feilen er $|f(x) - T_n f(x)| = |R_n f(x)|$.

$$|R_n f(x)| = \frac{(n+2)}{|c|^{n+3}} |x-1|^{n+1}$$

$c \in (1, 1.25)$. Da er $c \geq 1 \Rightarrow$

$$|R_n f(x)| \leq (n+2)\, |x-1|^{n+1}.$$

$x \in [1, 1.25] \Rightarrow |R_n f(x)| \leq (n+2) \left(\frac{1}{4}\right)^{n+1}$. (*)

Finn minste n s.a. $(n+2)\left(\frac{1}{4}\right)^{n+1} < 0.02$

n	$(n+2)\left(\frac{1}{4}\right)^{n+1}$
0	0.5
1	0.1875
2	0.0625
3	0.0195

Vi kan ta $N = 3$.

Eksamen 2020 Del 1 Flervalgsspørsmål.

Oppgave 1. $y'' - 3y' + 2y = 0$, $y(0)=1$, $y'(0)=0$.

Kar. lig. $r^2 - 3r + 2 = 0 = (r-1)(r-2) = 0$

$r_1 = 1$, $r_2 = 2$. $\Rightarrow$ $y(x) = Ce^{r_1 x} + De^{r_2 x} = Ce^{x} + De^{2x}$.

$y(0) = 1 = C + D$ (i)

$y'(x) = Ce^{x} + 2De^{2x}$

$0 = y'(0) = C + 2D$ (ii)

$D = 1 - C \Rightarrow 0 = C + 2(1-C) = -C + 2$, $C = 2$

$D = -1 \Rightarrow \boxed{y(x) = 2e^{x} - e^{2x}}$

Oppg. 2.

Opp. 2. $f(x) = x^2 - 1$. La $x_0 = 3$. Newtons metoden

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad f'(x) = 2x$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0^2 - 1)}{2x_0}$$

$$= 3 - \frac{(3^2-1)}{2.3} = 3 - \frac{8}{6} = 3 - \frac{4}{3} = \frac{5}{3}.$$

<u>Opp. 3</u>. Estimer andrederiverte til $f(x)$ i a:

$$(*) \quad \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

La $f(x) = x^3$ med $a = 1$. Finn tilnærmingen.

$$(*) = \frac{(1+h)^3 - 2 \cdot 1^3 + (1-h)^3}{h^2}$$

$$= \frac{(\cancel{1} + \cancel{3h} + 3h^2 + \cancel{h^3}) - \cancel{2} + (\cancel{1} - \cancel{3h} + 3h^2 - \cancel{h^3})}{h^2}$$

$$= 6. \checkmark$$

$$f'(x) = 3x^2 \Rightarrow f''(x) = 6x \Rightarrow f''(1) = 6.$$

Opp. 4. Newtonformen til andregradspolynomet som interpolerer $f(x) = x^2$ i punktene $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ er:

Newtonformen: $p(x) = c_0 + c_1(x - x_0) + c_2(x-x_0)(x-x_1)$
$= c_0 + c_1 x + c_2 x(x-1)$.

$p(0) = f(0) = 0$, $\quad p(0) = c_0 \Rightarrow c_0 = 0$

$p(1) = f(1) = 1^2 = 1$, $\quad p(1) = c_0 + c_1 \cdot 1$.

$\Rightarrow 1 = c_0 + c_1 \Rightarrow c_1 = 1$.

$p(2) = f(2) = 2^2 = 4$ $\quad p(2) = c_0 + 2c_1 + 2c_2$
$= 2 + 2c_2$

$\Rightarrow 4 = 2 + 2c_2 \Rightarrow c_2 = 1$.

$\Rightarrow p(x) = x + x(x-1)$. ✓

Oppgave 5 $\quad f(x) = \sin x \quad a = 0.$

Ønsker minst n slik at $|R_n f(x)| \leq 0.01$ for alle $x \in [0, 1/3]$.

Svaret $\quad R_n f(x) = \dfrac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!}$

Siden $f(x) = \sin x \Rightarrow |f^{(n+1)}(c)| \leq 1$, alle c

$$\Rightarrow |R_n f(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\leq \frac{1}{3^{n+1}} \frac{1}{(n+1)!} \quad (*)$$

Prøv med $n=0$, $n=1$, $n=2$. $\Rightarrow (*) < 0.01$.

Svaret er 2.