Del 2. Eksamen 2020.

Oppgave 1. Vis ved induksjon at

$$\sum_{k=1}^{n} (3k^2 + k) = n(n+1)^2,$$

for alle $n \geq 1$.

Svar. La P_n være utsagnet at $\sum_{k=1}^{n} (3k^2+k) = n(n+1)^2$.

(i) vis at P_1 er sant

(ii) vis at P_{n+1} er sant hvis antar at P_n er sant.

(i). La $n = 1 \Rightarrow V.S. = \sum_{k=1}^{1} (3k^2+k) = 3.1^2+1 = 3+1 = 4$

$H.S. = 1.(1+1)^2 = 2^2 = 4.$

(ii) P_{n+1}? $V.S. = \sum_{k=1}^{n+1} (3k^2+k)$, $H.S. = \underline{(n+1)(n+2)^2}$.

$$V.S. = \sum_{k=1}^{n} (3k^2+k) + 3(n+1)^2 + (n+1)$$

$$= \underbrace{n(n+1)^2}_{\text{p.g.a. } P_n \text{ er sant}} + 3(n+1)^2 + (n+1)$$

$$= (n+1)\left(n(n+1) + 3(n+1) + 1\right)$$

$$= (n+1)\left(n^2 + 4n + 4\right)$$

$$= (n+1)(n+2)^2. \qquad \text{QED.}$$

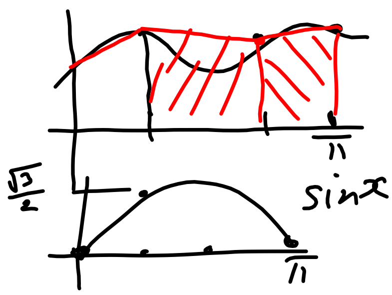
$\Rightarrow P_{n+1}$ er sant.

Oppgave 3. Approksimere $I = \int_0^{\pi} x^{1/2} \sin x \, dx$

a) Bruk trapesmetoden med tre delintervaller.

Svaret: $I \approx h\left(\frac{f(0)}{2} + f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right) + \frac{f(\pi)}{2}\right)$,

hvor $h = \frac{\pi}{3}$, $f(x) = x^{1/2} \sin x$.



$I \approx \frac{\pi}{3}\left(\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{\sqrt{2}}\right) \approx 2.2405$

$f\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)^{1/2} \sin\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)^{1/2} \frac{\sqrt{3}}{2} = \frac{\sqrt{\pi}}{2}$.

$f\left(\frac{2\pi}{3}\right) = \left(\frac{2\pi}{3}\right)^{1/2} \sin\left(\frac{2\pi}{3}\right) = \left(\frac{2\pi}{3}\right)^{1/2} \frac{\sqrt{3}}{2} = \frac{\sqrt{\pi}}{\sqrt{2}}$.

(b) Approksimere $\sin x$ med $T_5 \sin x$. om $a=0$.

Svar. $T_5 \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$.

$$I \approx \int_0^{\pi} x^{1/2} T_5 \sin x \, dx = \int_0^{\pi} x^{3/2} - \frac{x^{7/2}}{6} + \frac{x^{11/2}}{120} \, dx$$

$$= \left[\frac{2}{5} x^{5/2} - \frac{2}{9} \cdot \frac{x^{9/2}}{6} + \frac{2}{13} \cdot \frac{x^{13/2}}{120} \right]_{x=0}^{x=\pi}$$

$$= \frac{2}{5} \pi^{5/2} - \frac{2}{9} \cdot \frac{\pi^{9/2}}{6} + \frac{2}{13} \frac{\pi^{13/2}}{120}$$

$$\approx 2.7874$$

c) Bruke restleddet til å finne en øvre grense på feilen i (b). Hvis vi bruker T_n, hvilken n trenger vi for at feilen < 0.01. ?

Svar Feilen når vi bruker $T_n \sin x$ er :

$$\int_0^{\pi} x^{1/2} R_n \sin x \, dx$$

fordi $\int_0^{\pi} x^{1/2} \sin x \, dx - \int_0^{\pi} x^{1/2} T_n \sin x = \int_0^{\pi} x^{1/2} R_n \sin x \, dx$

$$R_n f(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}, \quad c \text{ er mellom } x \text{ og } a.$$

$$= \frac{\sin^{(n+1)}(c)\, x^{n+1}}{(n+1)!}$$

$$\left| \int_0^{\pi} x^{1/2} R_n \sin x \, dx \right| \le \int_0^{\pi} |x^{1/2}| \, |R_n \sin x| \, dx$$

$$\le \int_0^{\pi} x^{1/2} \frac{x^{n+1}}{(n+1)!} \, dx$$

$$\frac{1}{(n+1)!} \int_0^{\pi} x^{n+3/2} \, dx$$

$$= \frac{1}{(n+1)!} \left[\frac{x^{n+5/2}}{n+5/2} \right]_0^{\pi}$$

$$= \frac{1}{(n+1)!} \frac{\pi^{n+5/2}}{n+5/2}.$$

$n=5 \Rightarrow 0.9914.$

Finn n s.a. Feilen < 0.01.

Tilstrekkelig å velge $n=10$.

Oppgave 4 : Løs $x' = e^{-x}\cos t$, $x(0)=0$.

Svar : en separabel ligning :

$$e^{x} x' = \cos t$$

$$(e^{x})' = \cos t$$

Integrer :

$$e^{x} = \sin t + C$$

$x(0)=0 \Rightarrow$ $e^{0} = \sin 0 + C \Rightarrow 1+0=C,\ C=1.$

$$\Rightarrow e^{x} = \sin t + 1$$

ta ln : $x = \ln(\sin t + 1)$.

La $h=\frac{\pi}{4}$. bruk Eulers metode + Eulers m.p.m.

Eksakte, $x(\frac{\pi}{4}) = \ln(\sin\frac{\pi}{4} + 1) = \ln(\frac{1}{\sqrt{2}}+1)$
≈ 0.5348.

Euler $x_{k+1} = x_k + h f(t_k, x_k)$, $x_0 = x(0)$.

$t_0 = 0$, $t_1 = \frac{\pi}{4}$. $h=\frac{\pi}{4}$

$$f(t,x) = e^{-x}\cos t.$$

$$x_1 = x_0 + h f(t_0, x_0)$$
$$= 0 + \frac{\pi}{4} e^{-0}\cos(0) = \frac{\pi}{4}$$
$$= 0.7854$$

Euler M.P.
$$\begin{cases} x_{k+1/2} = x_k + \frac{h}{2} f(t_k, x_k) \\ x_{k+1} = \underline{x_k} + h f(t_{k+1/2}, x_{k+1/2}) \end{cases}$$

$t_{k+1} = t_k + h$ $\quad x_{1/2} = \cancel{x_0}^{0} + \frac{\pi}{8}\cdot 1 = \frac{\pi}{8}$

$t_{k+1/2} = t_k + \frac{h}{2}$ $\quad x_1 = \cancel{x_0}^{0} + \frac{\pi}{4}\cdot e^{-\pi/8}\cdot\cos(\frac{\pi}{8})$
$= 0.4900$

Her gir Eulers M.P. en bedre approks. , $t=\frac{\pi}{4}$.

Kan si at E.M. har feil $O(h)$

E.MPM ... $O(h^2)$

Da er det rimelig at MP er bedre her.