Suggestions/Additional exercises MAT-INF3100 Linear Optimization*

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We give some suggestions and additional exercises (from Vanderbei's book) in connection with the course.

Simplex algorithm: the basic technique etc.

1. For further drills in the basic computations of the simplex algorithm: you can simply make several examples yourself and work out the solution! And several exercises are in chapter 2 of Vanderbei's book. A good idea is to use the pivot tool on Vanderbei's course page (see link on the MAT-INF3100 course page) for checking your solution, and even to help in the numerical calculations of pivots. (But never forget how to make correct calculations by hand!!)

2. Work on (i) the primal simplex algorithm with "easy to see" initial basic solution, (ii) the primal simplex algorithm, phase I and II, (iii) the dual simplex algorithm, (iv) the network simplex algorithm (several exercises in Chapter 14, and you may vary the data and create more examples yourself!).

3. It is also useful to consider *general* algorithmic questions concerning the simplex algorithm, e.g. exercise 2.15, 2.18, 3.6, 4.10

Analysis of optimization problems.

1. This involves a study of general, or specific, linear programming models where the goal may be to (i) verify that a model is correct (represents the desired problem), (ii) show that the model has an optimal solution and interpret that solution, (iii) verify that the model has an optimal solution with some specific properties. This kind of activity may be hard, and requires some mathematical experience and knowledge. The compulsory project focuses on such analysis.

2. Investigate the following LP models and, for each, discuss if an optimal solution exists or if the problem is unbounded or infeasible. (i) $\max\{c^T x :$

*written by Geir Dahl, Dept. of Mathematics, University of Oslo

$x \geq O,\ x \in \mathbb{R}^n\}$, (ii) $\max\{c^T x : Ax = b,\ x \in \mathbb{R}^n\}$ (where A is an $m \times n$ matrix), (iii) $\max\{\sum_{j=1}^n c_j x_j : x \geq O,\ x \in \mathbb{R}^n\}$,

3. Some Vanderbei exercises along these lines are: 1.3, 2.19, 3.4, 5.13, 5.14 (here, replace (b) by : find explicit (analytically) an optimal solution of this problem), 6.6, 6.7, 7.9.

4. Concerning regression models (Chapter 12): exercise 12.1, 12.2, 12.3, 12.4, 12.10, 14.16, 14.17, 14.18.

Applications, LP modelling and using software.

1. Learn how to use OPL-CPLEX, here are some suggestions, but try out your own problems to model and solve!

2. Write a mod-file for a general problem in the form
$$\max\{c^T x : Ax \leq b,\ x \geq O\}.$$
Then write a corresponding dat-file and associate these in a project and solve some instances.

3. Write a OPL-CPLEX project for the linear approximation problem
$$\min\{\|Ax - b\|_1 : x \in \mathbb{R}^n\}$$
and solve some instances. You may compare to the corresponding least squares problem in a linear algebra text book. Then, do the same for the constrained linear approximation problem
$$\min\{\|Ax - b\|_1 : x \in \mathbb{R}^n,\ x \geq O\}.$$

4. Write a OPL-CPLEX problem for the problems in exercises 1.1 and 1.2.

5. Write a OPL-CPLEX problem for the diet problem in exercise 5.14.

6. The *assignment problem* is
$$\begin{array}{lll} \max & \sum_{ij} c_{ij} x_{ij} & \\ \text{s.t.} & & \\ & \sum_{j=1}^n x_{ij} = 1 & (i \leq n) \\ & \sum_{j=1}^n x_{ij} = 1 & (i \leq n) \\ & x_{ij} \geq 0 & (i, j \leq n) \\ & x_{ij} \text{ is an integer} & (i, j \leq n) \end{array}$$
and corresponds to finding the best (maximum weight) assignment of n jobs to n persons (or computers, or ...) when each person can do only one job. The matrix $C = [c_{ij}]$ contains the given weights. Read more about the problem (e.g. in Vanderbei (Chapter 15), or on the web, like wikipedia) and implement the model in OPL-CPLEX. A crucial feature here is that there always exists an optimal solution which is integral: we will see why in the network flow theory in the last part of the INF-MAT3370 course.

7. Write a OPL-CPLEX problem for the network flow problems in Exercises 15.3, 15.4, 15.9.

8. Look at the exercises above from regression models (Chapter 12) and implement some of them in OPL-CPLEX; also look at exercise 12.7.

9. If you are interested in learning some basic game theory, read Chapter 11 in Vanderbei, and try exercises 11.1, 11.2, 11.3, 116, 11.7

Further exercises

1. Consider the LP problem

$$
\begin{array}{llllllll}
\min & x_1 & - & 2x_2 & +x_3 & & & \\
\text{s.t.} & & & & & & & \\
 & x_1 & + & 2x_2 & + & 3x_3 & \leq & 10 \\
 & x_1 & - & x_2 & - & x_3 & = & 2 \\
 & & & \multicolumn{5}{r}{x_1 \text{ free},\ x_2 \geq 0,\ x_3 \leq -1}
\end{array}
$$

   Write this problem in our standard form ($\max\{c^T x : Ax \leq b, x \geq O\}$), and find the dual problem.

2. Consider the linear system $Ax \leq b$ where

$$
A = \begin{bmatrix} 3 & -1 \\ -1 & 0 \\ -1 & -1 \\ -4 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -1 \\ -1 \\ -4 \end{bmatrix}
$$

   Draw the solution set of $Ax \leq b$ in the plane and determine its extreme points.

3. Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Consider an inequality $c^T x \leq \alpha$ where $c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We say that the inequality $c^T x \leq \alpha$ is implied by $Ax \leq b$ if each $x_0 \in \mathbb{R}^n$ that satisfies $Ax \leq b$ also satisfies $c^T x \leq \alpha$. Explain how we can use linear optimization to decide whether $c^T x \leq \alpha$ is implied by $Ax \leq b$.

4. Recall that a polyhedron is a set P of the form $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ for some matrix A and vector b. How can be use LP to determine if a given polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is nonempty.