Compulsory project 1 in MAT-INF3100 Linear optimization, Spring 2015: Nonnegative vectors and mathematical finance

- **NOTE.** Send the project by *Thursday February 19 at 15:00* to Torkel Haufmann (torkelah@math.uio.no) in a single PDF file that you name "username.pdf" (your username!). Moreover, you should read the general information about compulsory projects at the course web page.

In this project we study nonnegative vectors in certain subspaces and how linear optimization plays a role for such questions. Moreover we shall relate these questions to some important problems in mathematical finance. No background in finance/economics is needed.

1 Nonnegative vectors and column spaces

A vector y is called *nonnegative*, written $y \geq O$ (where O is the zero vector), when $y_i \geq 0$ for each $i \leq n$. Let $\mathbb{R}^n_+$ be the set of nonnegative vectors in $\mathbb{R}^n$.

Let $A \in \mathbb{R}^{m\times n}$ be a real $m \times n$ matrix and let its columns be $a^1, a^2, \ldots, a^n$. Recall (from linear algebra) that the column space of A is

$$\text{Col}(A) = \{Ay : y \in \mathbb{R}^n\}$$

which is the same as the set of all linear combinations of the columns $a^1, a^2, \ldots, a^n$. This is a subspace of $\mathbb{R}^m$.

Consider now the linear transformation

$$T(x) = Ax.$$

Assume we have a "system" which takes a vector $x \in \mathbb{R}^n$ as input, and then produces the output $y = T(x) = Ax \in \mathbb{R}^m$. The goal is to study some questions concerning such a system; in particular when it produces nonnegative output. Moreover we shall relate these questions to mathematical finance.

The first questions are basic linear algebra. This gives you a chance, *which you should not miss (!)*, to recall some linear algebra!

- **Question 1:** *Assume that $m = n$ and that A is invertible. Show that any nonnegative output vector y can be produced. Actually, for any $y \in \mathbb{R}^m_+$ there exists a unique $x \in \mathbb{R}^n$ such that $y = T(x)$. Moreover, give a formula for this x.*

- **Answer:** If A is invertible, let $y \in \mathbb{R}^m_+$. Then if $x = A^{-1}y$, we have $y = Ax$ as desired. That this is unique is known from linear algebra, but we can note that if there exist x_1, x_2 such that $Ax_1 = Ax_2 = y$, then $A(x_1 - x_2) = y - y = 0$, and if $x_1 - x_2 \neq 0$ this contradicts the invertibility of A.

- **Question 2:** *Assume that $m \leq n$ and that* $\operatorname{rank}(A) = m$. *Show that for any $y \in \mathbb{R}^m_+$ there exists an $x \in \mathbb{R}^n$ such that $y = T(x)$. How can you find such an x?*

- **Answer:** As $\operatorname{rank}(A) = m$, A has m linearly independent columns and $\operatorname{Col}(A) = \mathbb{R}^m$. Hence for any $y \in \mathbb{R}^m_+$ there is an x so $y = Ax$, but in general this x is not unique. Such an x may be obtained by standard methods for solving systems of linear equations.

So we now know a large class of matrices that contain an arbitrary nonnegative vector in its column space. We shall study what happens when $\operatorname{rank}(A) < m$.

- **Question 3:** *Find an 2×2 matrix A such that the only nonnegative vector in* $\operatorname{Col}(A)$ *is the zero vector. Then, consider the matrix*

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ -1 & -2 \end{bmatrix}.$$

Find the columnspace of A and verify that O is its only nonnegative vector. Finally, modify a_{32} into another number (different from -2) such that the columnspace of the new A contains a nonzero, nonnegative vector y, and find such a vector y.

- **Answer:** First, consider the matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The column space of this matrix is spanned by $[1, -1]^T$, and it is clear that any nonzero vector in this subspace has at least one negative element.

For the second part, $\operatorname{Col}(A)$ is spanned by the columns of A:

$$\operatorname{Col}(A) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \right\}.$$

Now, if $y \in \operatorname{Col}(A)$ there exist x_1, x_2 such that

$$y = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + 3x_2 \\ -x_1 - 2x_2 \end{bmatrix}.$$

We split into three cases.

(i) If $x_1 = x_2$, $y = \left[0, x_1, -3x_1\right]^T$, which has a negative and a positive component if $x_1 \neq 0$, or is O if $x_1 = 0$.

(ii) If $x_1 > x_2$, the first component of y is positive. We will see that one of the other two is negative. Note that $-x_1 + x_2 < 0$, so $-2x_1 + 3x_2 = -2x_1 + 2x_2 + x_2 < x_2$. If $x_2 < 0$ we're done. If not, $x_2 \geq 0$ implies $x_1 > 0$, so $-x_1 - 2x_2 < 0$ and we're done.

(iii) If $x_2 > x_1$ the first component of y is negative, and we're done.

Finally, consider the matrix obtained by modifying a_{32} to be 2. Then

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

2 Why are these questions of interest?

Consider the situation above, with input vector x and output vector y, so $y = Ax$. Assume that x is an investment vector with one component for each investment possibility (at time t_0), say in n different assets (Norwegian: verdipapir, f.eks. aksje). x_j is the number of units we buy of asset j. Usually, $x_j \geq 0$. However, $x_j < 0$ is permitted; for instance, $x_j = -1$ means borrowing one asset (of asset j) at time t_0 and the value of this at time t_1 has to be paid back then. This means that you earn money if the value of the asset goes down (i.e., $a_{ij} < 0$ for state ω_i). The value of each asset after one time step (say a month), i.e., at time t_1, is unknown, so any model must capture this uncertainty. Let $\omega_1, \omega_2, \ldots, \omega_m$ be possible "states of the world" at time t_1 and

- let a_{ij} be the value of asset j under state ω_i at time t_1 minus the value of asset j at time t_0, i.e., the net profit.

These numbers may be organized into an $m \times n$ matrix $A = [a_{ij}]$. The numbers in column j of A give the net profit for asset j in the different states. Typically, this column contains both positive and negative numbers (and perhaps zeros): we may win or we may lose depending on the economic development. Note here that the matrix A completely specifies the random outcomes (so a_{ij} is not a random variable). However, the value (at time t_1) of asset j is a random variable with possible realizations $a_{1j}, a_{2j}, \ldots, a_{mj}$. We do not here specify the probability of each of the m states ω_i, but this is also used in financial modeling.

So, with this interpretation and when $y = Ax$, it means that y is the "net profit vector" under different states for the given investment vector x. (The vector x is usually called a *trading strategy* in mathematical finance).

It should be clear that finding an x such that the net profit $y = Ax$ is nonnegative (and nonzero) is an interesting task! If we succeed, it means that we do not lose money in any state, and, in at least one state, we gain money. This possibility, if it exists, is called an *arbitrage*.

By the way, note the beauty of this model: the financial market is simply specified by a matrix A; great!

- At this point, you might say: "This was interesting, but where is linear programming?"

OK, see the next section!

3 Connection to LP

Let A be an $m \times n$ matrix. Consider the LP problem

$$\begin{array}{ll} \max & \sum_{i=1}^{m} y_i \\ \text{subject to} & \\ & y = Ax \\ & y \geq O. \end{array} \tag{1}$$

The variable vector here is (x, y) where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ (viewed as column vectors). Note that we have nonnegativity constraints only on y.

- **Question 4:** *Show that the LP problem (1) is either unbounded or has an optimal solution with optimal value 0.*
- **Answer:** By the Fundamental Theorem of LP (Theorem 3.4 in Vanderbei), the problem either has an optimal solution or is infeasible or unbounded. $(x, y) = (0, 0)$ is always feasible in (1) with objective value 0, so the problem is always feasible.

  Now, if there exists a solution with greater objective value than 0, there is a pair (x, y) such that $y = Ax$, $y \geq 0$ and $e^T y > 0$, then for any $\lambda > 0$ we have $\lambda y \geq 0$, $\lambda y = A\lambda x$ and $e^T(\lambda y) = \lambda(e^T y)$. By increasing λ we obtain arbitrarily large solution values in (1), and hence the problem is unbounded. The conclusion follows.
- **Question 5:** *Consider the LP problem (1) where A represents a financial market. Show that the optimal value in (1) is zero if and only if the market has no arbitrage.*
- **Answer:** If there is a solution (x, y) to (1) where $y \neq 0$, this y is an arbitrage by definition, so this is possibly only if the market A has an arbitrage. On the other hand, if the market has no arbitrage, then any vector $y = Ax$ with $y \geq 0$ has to satisfy $y = 0$. Then the optimal objective function value in (1) has to be 0.
- **Question/task 6:** *Use OPL-CPLEX to implement the model (1). Write both a mod-file and a dat-file. Run your program on the matrix A in Question 3. Make a couple of other matrices and report your computations. Find an example where changing a single entry in the matrix makes all the difference when it comes to existence of arbitrage.*
- **Answer:** The code is located in the appendix, along with DAT files for the example market in Q3. OPL reports an optimal solution with objective value 0 when there is no arbitrage, and an unbounded problem when there is none.

  Reasonable other matrices would involve a larger selection of assets and states. As a small example, we consider the following matrix:

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -2 & 10 \\ -2 & 1 & -7 \\ 2 & -3 & -8 \end{bmatrix}.$$

Here there is no arbitrage, but changing a_{23} to -10 causes there to be.

4 A theorem in mathematical finance

It is a natural, and interesting, question to "explain" when an arbitrage possibility exists. One may argue that in any "natural market" it should not exist. Mathematically one therefore seeks a characterization of this existence question. We briefly discuss this.

Let again A be the matrix of a financial market. A *risk-neutral probability measure* is a vector z with positive components that sum to 1 such that the dot product of z and each column of A is zero. Mathematically, this means that $z^T A = O$, i.e., that $z \in \text{Nul}(A)^T$ (here Nul denotes the null space). Such a z may be viewed as a probability distribution on the set of possible states $\omega_1, \omega_2, \ldots, \omega_m$. Then z is a risk-neutral probability measure precicely when the *expected payoff* of each asset is zero.

Here is a basic theorem in mathematical finance:

Theorem 1 *(***Arbitrage theorem***) The market A has no arbitrage if and only if there is a risk-neutral probability measure z.*

This theorem may be proved using LP theory. Actually, it involves *LP duality theory* which we shall present later in the course. Here we only take a brief look at this. Recall first that

- $\sup(K)$ is the smallest upper bound of a set K of real numbers; if the set is unbounded above we write $\sup(K) = \infty$,
- $\inf(K)$ is the largest lower bound of K; if $K = \emptyset$ we write $\inf(K) = \infty$.

Define the two sets

$$S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = Ax,\ y \geq O\}$$
$$T = \{(z, w) \in \mathbb{R}^m \times \mathbb{R}^m : A^T z = O,\ z = w + e,\ w \geq O\}$$

where e denotes an all ones vector, $e = (1, 1, \ldots, 1)$ (again identified with the corresponding column vector).

From LP duality theory it follows that

$$\sup\{\sum_i y_i : (x, y) \in S\} = \inf\{0 : (z, w) \in T\}. \qquad (2)$$

Later you will learn how to derive this, so here you may just accept it! The infimum on the right-hand side may look a bit stange: it is the infimum of the constant function 0 (of (z, w)) over the set T. If T is empty, this infimum is ∞, otherwise it is 0.

- **Question 7:** *Prove Theorem 1 using (2).*
- **Answer:** Assume there is a $z \in T$. Define $z' = (1/e^T z)z$. Then $e^T z' = 1$, and $(z')^T A = O$ (as $z^T A = O$), so z' is a risk-neutral probability measure. If we have a risk-neutral probability measure z', let ϵ be the smallest component of z' (Recall $z' > 0$). Then define $z = (1/\epsilon)z'$. Now $z \geq e$,

and as $(z')^T A = O$ we have $z^T A = O$. In other words, $z \in T$. So T is non-empty if and only if there is a risk-neutral probability measure for the market A.

The supremum in (2) is the optimal value of the LP (1), so by Q5 we know that this is 0 if and only if there is no arbitrage (and ∞ otherwise). Then T is non-empty if and only if there is no arbitrage, and so there is a risk-neutral probability measure if and only if there is no arbitrage.

Good luck!

References

[1] G. Dahl. An Introduction to Convexity. Lecture notes, University of Oslo, 2014 (may be downloaded from course webpage).

[2] S.R. Pliska. *Introduction to Mathematical Finance: Discrete Time Models.* Blackwell Publishers, 1997.

[3] R. Vanderbei. *Linear programming: foundations and extensions.* Springer, Third edition, 2008.

OPL-CPLEX code

Model

```
/*********************************************
 * OPL 12.5 Model
 * Author: torkelah
 * Creation Date: 27. jan. 2015 at 11.35.16
 *********************************************/

int m = ...;
int n = ...;

range states = 1..n;
range assets = 1..m;

float A[states][assets] = ...;

dvar float+ y[states];
dvar float x[assets];

maximize sum(i in states) y[i];

subject to {
  forall (i in states)
    y[i] == sum(j in assets) A[i][j] * x[j];
}
```

Q3 Data file

```
/*********************************************
 * OPL 12.5 Data
 * Author: torkelah
 * Creation Date: 27. jan. 2015 at 11.38.03
 *********************************************/

n = 3;
m = 2;

A = [[1, -1],[-2,3],[-1,-2]];
```

Q3 No arbitrage

```
/*******************************************
 * OPL 12.5 Data
 * Author: torkelah
 * Creation Date: 27. jan. 2015 at 11.39.35
 *******************************************/

n = 3;
m = 2;

A = [[1,  -1],[-2,3],[-1,2]];
```

Other example

```
/*******************************************
 * OPL 12.5 Data
 * Author: torkelah
 * Creation Date: 30. jan. 2015 at 10.47.05
 *******************************************/

n = 4;
m = 3;

A = [[-1,   3,   5],
     [ 1,  -2,  10],
     [-2,   1,  -7],
     [ 2,  -3,  -8]];
```