Compulsory Project 2
MAT-INF3100 Linear optimization, Spring 2015

- **NOTE.** Hand in your assignment using Devilry (devilry.ifi.uio.no) in a single PDF file that you name "username.pdf" (your username!). You must also attach the source code you wrote for Problem 3 (You are expected to discuss it as well as hand it in). Moreover, you should read the general information about compulsory projects at the course web page. Your submission should be written by yourself and reflect your own understanding.

Problem 1

1.1

Write down the dual problems for the following LP problems:

$$\begin{array}{ll} \text{min} & 3x_1 + 5x_2 - x_3, \\ \text{subject to} & \\ & x_1 - x_2 + x_3 \leq 3, \\ & 2x_1 - 3x_2 \leq 4, \\ & x_1, x_2 \geq 0. \end{array} \tag{1}$$

and

$$\begin{array}{ll} \text{max} & 3x_1 + 2x_2, \\ \text{subject to} & \\ & 4x_1 + 2x_2 \leq 16, \\ & x_1 + 2x_2 \leq 8, \\ & x_1 + x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{array} \tag{2}$$

1.2

Write (1) and its dual problem in matrix form. Do the same for (2).

1.3

Show that $x^\star = (3, 2)$ is feasible for the primal problem (2) and $y^\star = (1/2, 0, 1)$ is feasible for the corresponding dual problem. Moreover, show that $x^\star$ is in fact the optimal solution of (2).

Problem 2

Consider the function

$$S(x,y) = f(x) - xy + g(y), \qquad (x,y) \in \mathbf{R}^2, \qquad \mathbf{R} = (-\infty, \infty),$$

where f is a strongly concave function and g is a strongly convex function.

A function h is called strongly convex if $h''(y) > 0$ for all $y \in \mathbf{R}$, $h'(y) \to -\infty$ as $y \to -\infty$, and $h'(y) \to \infty$ as $y \to \infty$. A function is strongly concave if $-h$ is strongly convex.

2.1

Show that the gradient $\nabla S = (\partial S/\partial x, \partial S/\partial y)$ vanishes at a unique point $(x^\star, y^\star) \in \mathbf{R}^2$.

2.2

With the stationary point $(x^\star, y^\star) \in \mathbf{R}^2$ given above, show that

$$\max_{x\in\mathbf{R}} \min_{y\in\mathbf{R}} S(x,y) = S(x^\star, y^\star), \qquad \min_{y\in\mathbf{R}} \max_{x\in\mathbf{R}} S(x,y) = S(x^\star, y^\star).$$

Hint: Verify second order derivative tests for both inner and outer optimisations.

Problem 3

Assume we have an LP problem on standard form:

$$\max\{c^T x : Ax \leq b,\ x \geq O\}$$

For simplicity we only work with matrices $A \geq 0$ and assume that $b \geq 0$, so $x = 0$ is a feasible solution (A is an m times n matrix, and b, c are appropriately sized). Given this, implement the following two algorithms:

- **Algorithm S**: the primal simplex algorithm (Section 6.2 in the book)
- **Algorithm I**: the path-following interior point algorithm (Figure 18.1 in the book, see chapters 17, 18 and 19 for various details)

The choice of language is up to you, but we recommend MATLAB (or perhaps Python). Hand in your code (in separate files) and a test run for both algorithms, using the data file on the course web page as a test. Print a line or two of useful information every iteration. Discuss your results.

Some further points:

- The optimal value in the example problem is 212.6539.
- When solving linear systems of equations $Vx = d$ do not use the `inv(V)` command – rather use the backslash operator, i. e. `x = V \ d`.

- The major part of the work for algorithm I is solving the *KKT-system* (Karush-Kuhn-Tucker system, the optimality conditions for the barrier problem) to find the steps Δx, Δy, Δz and Δw. This can be done in different ways, but we suggest using the *normal equations in primal form*, see start of section 19.2, eq. (19.9) for finding Δy, then find Δx etc.

- If you are using MATLAB and have downloaded the `.mat` file from the webpage, you can use the command `load('lptestproblemoblig.mat')` to have the correct A, b, c appear in your MATLAB workspace.

- The test problem is quite large, so it might be a good idea to develop against a simpler problem at first – see for example pages 96-101 in Vanderbei (2nd ed.).

Good luck!