LP. Lecture 2. Chapter 2: the simplex method, cont.

- initialization
- two phases
- unbounded solution
- geometry

Repetition

- LP problem
- feasible solution, optimal solution
- dictionary
- basis, basic variable and nonbasic variable
- pivot
- see example from Lecture 1

Initialization

LP problem:

$$
\begin{array}{lrll}
\max & \sum_{j=1}^{n} c_j x_j & & \\
\text{subj.to} & & & \\
& \sum_{j=1}^{n} a_{ij} x_j & \leq b_i & \text{for } i = 1, \ldots, m \\
& x_j & \geq 0 & \text{for } j = 1, \ldots, n.
\end{array}
$$

Introduce slack variables and obtain the dictionary:

$$
\begin{array}{rcll}
\eta & = & \sum_{j=1}^{n} c_j x_j & \\
w_i & = & b_i - \sum_{j=1}^{n} a_{ij} x_j & \text{for } i = 1, \ldots, m
\end{array}
$$

So if $b_i \geq 0$ for all $i \leq m$ we find an initial feasible solution by letting

$$
\begin{array}{ll}
w_i = b_i & \text{for all } i \leq m \text{ and} \\
x_j = 0 & \text{for all } j \leq n.
\end{array}
$$

Problem: what if some of the b_i's are negative?

Solution:

- consider finding a first feasible solution as a new problem
- and this problem may be written as an LP problem !!!!
- the new LP problem is called the Phase I problem
- fortunately: it is easy to find an initial feasible solution of the phase I problem !

Phase I problem:

$$\begin{array}{lrll} \max & -x_0 & & \\ \text{subject to} & & & \\ & \sum_{j=1}^{n} a_{ij}x_j - x_0 & \leq b_i & \text{for } i = 1, \ldots, m \\ & x_j & \geq 0 & \text{for } j = 0, 1, \ldots, n. \end{array}$$

Note: this is also an LP problem. We denote it by LP-I.

The Phase I problem:

- same variables as before, but one extra variable x_0
- the original objective function is replaced by $-x_0$. We want to *maximize* $-x_0$. This is equivalent to *minimize* x_0.
- the constraints are "almost as before" and they are

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + x_0$$

Interpretation of the Phase I problem:

The Phase I problem determines if the original problem has a feasible solution (which may be highly nontrivial). And, if the answer is positive, then it also find a feasible solution. More on this:

- think about x_0 as an increase of each right-hand side b_i.

- If we find a feasible solution of LP-I where this increase is 0 ($x_0 = 0$), then we have

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i + x_0 = b_i$$

  so then $(x_1, \ldots, x_n)$ is a feasible solution in the original LP problem! Great!
- If there is no feasible solution in LP-I with $x_0 \leq 0$, then the original problem does not have any feasible solution either. Why?

Initialization, example

An LP problem (with at least one negative b_i):

$$
\begin{array}{lrcrcr}
\max & -2x_1 & - & x_2 & & \\
\text{subj. to} & & & & & \\
 & -x_1 & + & x_2 & \leq & -1 \\
 & -x_1 & - & 2x_2 & \leq & -2 \\
 & & & x_2 & \leq & 1 \\
 & & & & & x_1, x_2 \geq 0 \ .
\end{array}
$$

This gives the Phase I problem:

$$
\begin{array}{lrcrrcr}
\max & & & & -x_0 & & \\
\text{subj. to} & & & & & & \\
 & -x_1 & + & x_2 & -x_0 & \leq & -1 \\
 & -x_1 & - & 2x_2 & -x_0 & \leq & -2 \\
 & & & x_2 & -x_0 & \leq & 1 \\
 & & & & & & x_0, x_1, x_2 \geq 0 \ .
\end{array}
$$

This is the frist dictionary. Which is nonfeasible!!? (Why?)

$$
\begin{array}{rrrrrrrrr}
\xi & = & & & & & - & x_0 \\
\hline
w_1 & = -1 & + & x_1 & - & x_2 & + & x_0 \\
w_2 & = -2 & + & x_1 & + & 2x_2 & + & x_0 \\
w_3 & = 1 & & & - & x_2 & + & x_0
\end{array}
$$

But it can be converted into a feasible dictionary by a pivot! We let x_0 enter the basis (interpretation: we need a surplus on the right-hand side) and let the variable which is the most negative leave the basis. This is w_2 here.

Result:

$$
\begin{array}{rrrrrrrr}
\xi & = -2 & + & x_1 & + & 2x_2 & - & w_2 \\
\hline
w_1 & = 1 & & & - & 3x_2 & + & w_2 \\
x_0 & = 2 & - & x_1 & - & 2x_2 & + & w_2 \\
w_3 & = 3 & - & x_1 & - & 3x_2 & + & w_2
\end{array}
$$

This is a feasible dictionary, and the corresponding basic solution is

$$w_1 = 1, x_0 = 2, w_3 = 3 \quad \text{and the other variables are zero.}$$

What next? Since we have a feasible dictionary (i.e., a dictionary with a basic feasible solution) we proceed with the simplex algorithm from this solution. So: pivot until we have found an optimal solution. (We use floating point numbers to increase readability).

1. iteration: x_2 into the basis and w_1 out, which gives:

$$\begin{array}{rcrcrcrcr}
\xi & = & -1.33 & + & x_1 & - & 0.67w_1 & - & 0.33w_2 \\
\hline
x_2 & = & 0.33 & & & - & 0.33w_1 & + & 0.33w_2 \\
x_0 & = & 1.33 & - & x_1 & + & 0.67w_1 & + & 0.33w_2 \\
w_3 & = & 2 & - & x_1 & + & w_1 & &
\end{array}$$

2. iteration: x_1 into the basis and x_0 out, which gives:

$$
\begin{array}{rclllll}
\xi & = & 0 & - & x_0 & & & & \\
\hline
x_2 & = & 0.33 & & & - & 0.33w_1 & + & 0.33w_2 \\
x_1 & = & 1.33 & - & x_0 & + & 0.67w_1 & + & 0.33w_2 \\
w_3 & = & 0.67 & + & x_0 & + & 0.33w_1 & - & 0.33w_2
\end{array}
$$

We see that this dictionary is optimal! So we have solved the Phase I problem.

Since the optimal value is 0, there is a feasible solution in the original LP problem. Namely: $x_1 = 4/3, x_2 = 1/3$. Check this!

Furthermore, we can write down a feasible dictionary for the original LP problem by

- removing x_0 from the optimal dictionary in Phase I, and
- reintroduce the original objective function:

$$\begin{array}{rcrcrcr}
\eta & = & -3 & - & w_1 & - & w_2 \\
\hline
x_2 & = & 0.33 & - & 0.33w_1 & + & 0.33w_2 \\
x_1 & = & 1.33 & + & 0.67w_1 & + & 0.33w_2 \\
w_3 & = & 0.67 & + & 0.33w_1 & - & 0.33w_2
\end{array}$$

Now we have a feasible dictionary (and a basic feasible solution) and we solve this problem using the simplex algorithm. We call this the Phase II problem.

By chance, this dictionary is already optimal, so no pivot was necessary. Normally several pivots are required to solve the Phase II problem.

Initialization

Summary:

- the method in the example may be used in general.
- Phase I: solve the Phase I problem to find, if possible, an initial feasible solution (actually a basic feasible solution). This is a feasible starting point for the next phase.
- Phase II: solve the Phase II problem using the solution from Phase I as the starting point. We then find, using the simplex algorithm, an optimal solution of the original LP problem or, possibly, and unbounded solution.

Unbounded solution

Next topic: some problems have an unbounded value!

Look closer at a pivot. Recall:

- an index k is moved from N to B (x_k is entering variable; new basic variable because it increases η,
- another index l is moved from B to N (x_l is leaving variable; this variable leaves the basis because it becomes zero, and
- we find a new feasible dictionary from the previous one by row operations.

Possible problem: it could be that when the ingoing variable x_k is increased, then none of the basic variables become zero! Example:

$$\begin{array}{ccccccc}
\eta & = & 2 & + & 3x_4 & - & x_5 \\
\hline
x_1 & = & 1 & + & x_4 & + & x_5 \\
x_2 & = & 2 & + & 5x_4 & + & x_5 \\
x_3 & = & 0 & & & - & x_5
\end{array}$$

We wish to take x_4 into the basis, and see that no basic variable becomes zero.

Conclusion: may increase x_4 without bounds and thereby obtain arbitrarily large value on the objective function η. We then say that the problem is unbounded, and that the optimal "value" is ∞. Further, we see that if x moves along the ray

$$(x_1, x_2, x_3, x_4, x_5) = (1, 2, 0, 0, 0) + (1, 5, 0, 1, 0)x_4$$

where $x_4 \geq 0$, then $\eta \to \infty$.

What can be said in general?
Let x_k be the variable that enters the basis. If all the coefficients (in the dictionary) of x_k are nonnegative, then the same thing as above will happen.
So:

- unbounded value, and
- we find a ray where η goes towards infinity.

Geometry

Geometry for LP in two variables (figure!):

- feasible set: polyhedron P.
- feasible basic solution: vertex.
- pivot: move along an edge between two neighbor vertices.
- the level set $\{x : c^T x = \alpha\}$: line orthogonal to c, translate this line until it becomes a tangent to P.
- the similar geometry for several variables: see later (convexity).

Final comments

When we solve an LP problem the following possibilities are present:

1. the problem has no feasible solution. If so, this is determined in Phase I.
2. the problem has a feasible solution, but no optimal solution because the problem is unbounded.
3. the problem is feasible but not unbounded, and we terminate with and optimal solution

And these are in fact *all* possibilities

But the reason for this is nontrivial. We need to prove that the simplex algorithm always terminates.

- We have so far assumed that all basic variable are positive, but we have not discussed what happens if at least one of them is zero!
- This is the topic we consider in Lecture 3.