UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 10, 2012.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let $\mathcal{L}$ be a countable first-order language.

Problem 1

What does it means that a set of $\mathcal{L}$-sentences is consistent?

Problem 2

State the Deduction Theorem for first-order logic.

Problem 3

Let Σ be a consistent set of $\mathcal{L}$-formulas, and let ϕ be an $\mathcal{L}$-sentence. Prove that a least one of the sets $\Sigma \cup \{\phi\}$ and $\Sigma \cup \{\neg\phi\}$ is consistent.

Problem 4

Let $\{\Sigma_i\}_{i\in\mathbb{N}}$ be an infinite sequence of consistent sets of $\mathcal{L}$-sentences such that $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$. Prove that the set $\bigcup_{i\in\mathbb{N}} \Sigma_i$ is consistent.

Problem 5

Show that any consistent set of $\mathcal{L}$-sentences can be extended to a maximal consistent set.

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Part II

Let $\mathcal{L}$ be the first-order language $\{c_0, S\}$ where S is a binary relation symbol and c_0 is a constant symbol. We will write the relation symbol in infix notation. Let T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(A_1) $\forall x[\neg xSx]$

(A_2) $\forall x[\neg xSc_0]$

(A_3) $\forall xyz[\, xSy \wedge xSz \;\rightarrow\; y = z\,]$

(A_4) $\forall xyz[\, ySx \wedge zSx \;\rightarrow\; y = z\,]$

(A_5) $\forall x[\, x \neq c_0 \;\rightarrow\; \exists y[ySx]\,]$.

Problem 6

Give a model for T where the universe is $\{0, 1, 2, 3\}$.

Problem 7

Prove that $T \vdash xSy \rightarrow y \neq x$ by giving a T-derivation.

Let $x_1, x_2, \ldots$ be variables. For $n \geq 1$, we define the formula ϕ_n by

- $\phi_1 \equiv c_0 S x_1$
- $\phi_{n+1} \equiv \phi_n \wedge x_n S x_{n+1}$.

Lemma (A). Let i, j, n be natural numbers such that $1 \leq n$ and $i, j \leq n$ and $i \neq j$. Then

$$T \vdash \phi_n \;\rightarrow\; x_i \neq x_j \;.$$

Problem 8

Name the axioms of T that will be needed in a proof of Lemma (A), and name the axioms of T that will not needed in a proof of Lemma (A). (You are not asked to prove the lemma.)

We extend the language $\mathcal{L}$ by infinitely many constants $c_1, c_2, \ldots$. For each $i \in \mathbb{N}$, we extend the theory T by the axiom $c_i S c_{i+1}$. An $\mathcal{L}$-formulas is atomic if it is in form $t_1 S t_2$ or in the form $t_1 = t_2$ (for some terms t_1 and t_2).

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Problem 9

Let α be a variable-free atomic formula. Show that $T \vdash \alpha$ or $T \vdash \neg\alpha$.

Let $\mathfrak{N}$ be the $\mathcal{L}$ structure where the universe is the set of natural numbers $\mathbb{N}$ and

- $c_i^{\mathfrak{N}} = i$ for any $i \in \mathbb{N}$
- $S^{\mathfrak{N}} = \{\langle i, i+1\rangle \mid i \in \mathbb{N}\}$.

We define a Σ-formula inductively by

- α and $\neg\alpha$ are Σ-formulas if α is an atomic formula
- $(\alpha \wedge \beta)$ is a Σ-formula if α and β are Σ-formulas
- $(\alpha \vee \beta)$ is a Σ-formula if α and β are Σ-formulas
- $(\exists x)(\alpha)$ is a Σ-formula if α is a Σ-formula and x is a variable.

Theorem (B). Let $\psi(x_1, \ldots, x_m)$ be a Σ-formula with free variables $x_1, \ldots, x_m$, and let $t_1, \ldots, t_m$ be variable-free $\mathcal{L}$-terms. Then,

$$\mathfrak{N} \models \psi(t_1, \ldots, t_m) \quad \Rightarrow \quad T \vdash \psi(t_1, \ldots, t_m) \; .$$

Problem 10

Prove Theorem (B).

Problem 11

Show that theory T is incomplete, that is, show that there exists an $\mathcal{L}$-formula θ such that $T \nvdash \theta$ and $T \nvdash \neg\theta$.

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