UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Tuesday, December 10, 2013.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let $\mathcal{L}$ be the language $\{R\}$ where R is a binary relation symbol.

Problem 1

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that

$$\mathfrak{A} \models \forall x \exists y [Rxy] \to \exists y \forall x [Rxy] .$$

Problem 2

Give an $\mathcal{L}$-structure $\mathfrak{B}$ such that

$$\mathfrak{B} \not\models \forall x \exists y [Rxy] \to \exists y \forall x [Rxy] .$$

Problem 3

Is $\forall x \exists y [Rxy] \to \exists y \forall x [Rxy]$ a valid formula? Justify your answer.

Problem 4

Prove that

$$\vdash \forall x \forall y [Rxy] \to \forall x [Rxx]$$

by giving a deduction.

Problem 5

Is $\forall x \forall y [Rxy] \to \forall x [Rxx]$ a valid formula? Justify your answer.

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Part II

Let $\mathcal{L}$ be a first-order language, and let $\mathcal{L}^\exists$ be $\mathcal{L}$ extended with fresh (new) constant symbols $c_1^\psi, \ldots, c_n^\psi$ for each $\mathcal{L}$-sentence of the form $\exists x_1 \ldots x_n[\psi(x_1, \ldots, x_n)]$. When Σ is a set of $\mathcal{L}$-formulas, let $\Sigma^\exists$ denote the least set of $\mathcal{L}^\exists$-formulas such that

(1) $\Sigma^\exists$ contains all formulas in Σ

(2) for each $\mathcal{L}$-sentence of the form $\exists x_1 \ldots x_n[\psi(x_1, \ldots, x_n)]$, the set $\Sigma^\exists$ contains the $\mathcal{L}^\exists$-formula
$$\exists x_1 \ldots x_n[\psi(x_1, \ldots, x_n)] \rightarrow \psi(c_1^\psi, \ldots, c_n^\psi) .$$

Note that (2) requires $\exists x_1 \ldots x_n[\psi(x_1, \ldots, x_n)]$ to be a sentence, that is, there are no free variables in $\psi(x_1, \ldots, x_n)$ except $x_1, \ldots, x_n$. Furthermore, note that the constant symbols $c_1^\psi, \ldots, c_n^\psi$ will occur in one, and only one, formula in the set $\Sigma^\exists$.

Let $\mathcal{L}_0$ be the language $\{P, Q\}$ where P and Q are unary relation symbols. Let
$$\Sigma_0 \quad = \quad \{\, \exists xy[Px \wedge Qy] \, , \, \forall x[\neg(Px \wedge Qx)] \,\} \, .$$

Problem 6

Give a $\Sigma_0^\exists$-deduction of $\exists x[Px]$.

Problem 7

Give a $\Sigma_0^\exists$-deduction of $\exists xy[x \neq y]$. Try to give a full deduction. Do not argue that such a deduction exists by referring to lemmas and theorems in Leary's book.

Problem 8

Let ξ be the formula $\exists x \exists y[Px \wedge Py \wedge x \neq y]$. Explain why $\Sigma_0 \nvdash \xi$. Explain why $\Sigma_0 \nvdash \neg\xi$.

Problem 9

Let $\mathcal{L}$ be a first-order language, and let Σ be a set of $\mathcal{L}$-formulas. Prove that Σ has a model if, and only if, $\Sigma^\exists$ has a model.

Problem 10

Let $\mathcal{L}$ be a first-order language, let Σ be a set of $\mathcal{L}$-formulas, and let ϕ be an $\mathcal{L}$-sentence. Prove that
$$\Sigma \vdash \phi \quad \Leftrightarrow \quad \Sigma^\exists \vdash \phi \, .$$

You may use the Deduction Theorem and the Completeness Theorem.

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Problem 11

Give a Σ_0-deduction of $\exists xy[x \neq y]$.

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