UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Tuesday, November 29, 2016.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let P and Q be unary relation symbols, and let a and b be constant symbols. Let $\mathcal{L}$ be the language $\{a, b, P, Q\}$. Let Σ be the set of $\mathcal{L}$-formulas given by

$$\Sigma \;=\; \{\, Pa \,,\, \neg Pb \,,\, Qb \,,\, \forall x[\, Px \rightarrow Qx \,]\,\} \,.$$

Problem 1

Give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$.

Problem 2

Give an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \not\models \Sigma$.

Problem 3

Give a Σ-deduction of Qa. Give a full deduction. Name all the logical axioms and inference rules involved in the deduction.

Problem 4

Give a Σ-deduction of $\neg\forall x[\, Qx \rightarrow Px \,]$. Give a full deduction. Name all the logical axioms and inference rules involved in the deduction.

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Problem 5

What does it mean that a set of $\mathcal{L}$-formulas is consistent? (Give the definition.) Is the set $\Sigma \cup \{ \forall x[Px] \}$ consistent? Is the set $\Sigma \cup \{ \forall x[\neg Px] \}$ consistent? Justify your answers.

Problem 6

Use the Soundness Theorem to prove that $\Sigma \nvdash \forall x[Px \vee Qx]$.

Part II

Problem 7

Let $\mathcal{L}$ be a first-order language without constant symbols and functions symbols (thus, $\mathcal{L}$ contains only relation symbols). Let $n \in \mathbb{N}$, let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be variables, and let $\phi(z_1, \ldots, z_n)$ be an $\mathcal{L}$-formula with free variables $z_1, \ldots, z_n$. Prove that

$$\vdash (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (\phi(x_1, \ldots, x_n) \rightarrow \phi(y_1, \ldots, y_n)) \ .$$

Use induction on the structure of the formula ϕ. You will need the axiom scheme

$$(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (Rx_1 \ldots x_n \rightarrow Ry_1 \ldots y_n) \ . \tag{E3}$$

Note that (E3) requires that R is a relation symbol. Note that ϕ does not contain constant symbols and function symbols.

Part III

Recall that $\mathfrak{A} \simeq \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Recall that $\mathfrak{A} \equiv \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. Recall that $\mathcal{L}_{NT}$ is the language $\{0, S, +, \cdot, E, <\}$. Recall that $\mathfrak{N}$ is the standard $\mathcal{L}_{NT}$-structure.

Let Σ be a set of $\mathcal{L}_{NT}$-sentences such that $\mathfrak{N} \models \Sigma$. We consider four properties the set Σ might possess:

(1) for every $\mathcal{L}_{NT}$-structure $\mathfrak{A}$, we have

$$\mathfrak{A} \models \Sigma \ \Rightarrow \ \mathfrak{A} \equiv \mathfrak{N}$$

(2) for every $\mathcal{L}_{NT}$-sentence ϕ, we have

$$\mathfrak{N} \models \phi \ \Rightarrow \ \Sigma \vdash \phi$$

(3) for every $\mathcal{L}_{NT}$-sentence ϕ, we have

$$\Sigma \cup \{\phi\} \nvdash \bot \ \Rightarrow \ \mathfrak{N} \models \phi$$

(4) for every $\mathcal{L}_{NT}$-structure $\mathfrak{A}$, we have

$$\mathfrak{A} \models \Sigma \ \Rightarrow \ \mathfrak{A} \simeq \mathfrak{N} \ .$$

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Problem 8

Prove that (1), (2) and (3) are equivalent.

Problem 9

Does (4) follow from (1)? Does (1) follow from (4)? Justify your answers.

Problem 10

Does there exist a set Σ of $\mathcal{L}_{NT}$-sentences such that $\mathfrak{N} \models \Sigma$ and (4) holds? Does there exist a set Σ of $\mathcal{L}_{NT}$-sentences such that $\mathfrak{N} \models \Sigma$ and (1) holds? Does there exist a *recursive* set Σ of $\mathcal{L}_{NT}$-sentences such that $\mathfrak{N} \models \Sigma$ and (1) holds? Justify your answers.

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