UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Friday, December 15, 2017.

Examination hours: 9:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let R and S be a unary relation symbols, and let a be a constant symbol. Let $\mathcal{L}$ be the language $\{a, R, S\}$.

Problem 1

State the Soundness Theorem for first-order logic. State the Completeness Theorem for first-order logic.

Problem 2

Let ϕ be an $\mathcal{L}$-formula such that $\nvdash \phi$. Explain why there exists an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models \phi$. Give a brief answer.

Problem 3

Below you will find six $\mathcal{L}$-formulas $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$. For each formula ϕ_i, we either have $\vdash \phi_i$ or $\nvdash \phi_i$. If $\vdash \phi_i$, you should give a detailed deduction of ϕ_i (name all the axioms and inference rules involved in the deduction). If $\nvdash \phi_i$, you should give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models \phi_i$.

- $\phi_1 :\equiv Ra \rightarrow (Sa \rightarrow Ra)$
- $\phi_2 :\equiv \exists x[Rx] \rightarrow \exists x[Sx \rightarrow Rx]$
- $\phi_3 :\equiv \forall x[Rx] \rightarrow \forall x[Sx \rightarrow Rx]$
- $\phi_4 :\equiv \exists x[Sx \rightarrow Rx] \rightarrow \exists x[Rx]$
- $\phi_5 :\equiv \forall x[Sx \rightarrow Rx] \rightarrow \forall x[Rx]$

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- $\phi_6 :\equiv (\forall x[Rx] \rightarrow Ra) \ \vee \ (\forall x[Sx \rightarrow Rx] \rightarrow \forall x[Rx])$

Part II

Let $<$ be a binary relation symbol, let S be unary function symbols, and let 0 be a constant symbol. Let $\mathcal{L}$ be the language $\{0, S, <\}$. Let T be the $\mathcal{L}$-theory where we have the following non-logical axioms:

(T_1) $\forall x[\neg Sx = 0]$

(T_2) $\forall xy[Sx = Sy \rightarrow x = y]$

(T_3) $\forall x[\neg Sx = x]$

(T_4) $\forall x[\neg x < 0]$

(T_5) $\forall xy[x < Sy \leftrightarrow (x < y \vee x = y)]$

Problem 4

Prove that the axiom T_3 is independent of the other axioms of T, that is, prove that

$$\{T_1, T_2, T_4, T_5\} \nvdash T_3 \quad \text{and} \quad \{T_1, T_2, T_4, T_5\} \nvdash \neg T_3 \ .$$

Let $\phi(x)$ be any $\mathcal{L}$-formula (as usual $\phi(t)$ denotes $\phi(x)$ where every free occurrence of the variable x is replaced by the term t). We will consider three axiom schemes.

The scheme of Zero Intolerance. This is the scheme

$$\forall x[\phi(Sx)] \rightarrow \forall x[\phi(x)] \tag{Z}$$

The theory T_Z is the theory T extended by this axiom scheme.

The scheme of Pseudo Induction. This is the scheme

$$(\,\phi(0) \ \wedge \ \forall x[\phi(Sx)]\,) \ \rightarrow \ \forall x[\phi(x)] \tag{P}$$

The theory T_P is the theory T extended by this axiom scheme.

The scheme of Induction. This is the scheme

$$(\,\phi(0) \ \wedge \ \forall x[\phi(x) \rightarrow \phi(Sx)]\,) \ \rightarrow \ \forall x[\phi(x)] \tag{I}$$

The theory T_I is the theory T extended by this axiom scheme.

Problem 5

Give a T_Z-deduction of $\neg 0 = 0$. Give a full deduction. Name all the axioms and inference rules involved in the deduction.

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Problem 6

Explain why we have $T_Z \vdash \theta$ for every $\mathcal{L}$-formula θ.

Problem 7

Prove that

$$T_P \vdash \forall x[\, x = 0 \;\vee\; \exists y[Sy = x]\,] \ .$$

Sketch a T_P-deduction of the formula. Name all the non-logical axioms involved in the deduction.

Problem 8

Prove that

$$T \not\vdash \forall x[\, x = 0 \;\vee\; \exists y[Sy = x]\,] \ .$$

Problem 9

Let θ be any $\mathcal{L}$-formula. Prove that $T_P \vdash \theta$ implies $T_I \vdash \theta$.

Problem 10

Does it exist an $\mathcal{L}$-formula η such that $T_I \vdash \eta$ and $T_P \not\vdash \eta$? Justify your answer. If you can, prove that your answer is correct.

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