UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Tuesday, December 1, 2020.

Examination hours: 15:00 – 19:00.

This problem set consists of 4 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I

Let P and Q be unary relation symbols. Let a, b and c be constant symbols. Let f and g be unary function symbols. Furthermore, x and y denote variables.

Problem 1 *(weight 10 %)*

Let $\Sigma = \{\, \forall x[\, Px\,]\,\}$. Give a full Σ-deduction of $\forall x[\, Qc \to Px\,]$. Name all the logical axioms and inference rules involved in the deduction.

Problem 2 *(weight 10 %)*

Let $\mathcal{L}$ be the language $\{\, P, Q, c\,\}$ Give an $\mathcal{L}$ structure $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x[\, Qc \to Px\,]$ and $\mathfrak{A} \not\models \forall x[\, Px\,]$, Explain briefly why we have $\{\, \forall x[\, Qc \to Px\,]\,\} \nvdash \forall x[\, Px\,]$.

Problem 3 *(weight 10 %)*

Ten Questions: Answer each question with a YES or a NO (and nothing else). If you do not answer a question, your answer to that question will be considered as wrong.

1. Is the set $\{\, a \neq b\, ,\, f(a) = f(b)\,]\,\}$ consistent?
2. Is the set $\{\, a = b\, ,\, f(a) \neq f(b)\,]\,\}$ consistent?
3. Is the set $\{\, f(a) = g(b)\, ,\, g(b) = f(c)\, ,\, g(f(a)) \neq g(f(c))\,]\,\}$ consistent?
4. Is the set $\{\, \forall xy[\, f(x) \neq f(y)\,]\, ,\, \exists xy[\, f(x) \neq f(y)\,]\,\}$ consistent?
5. Is the set $\{\, \forall xy[\, f(x) = f(y)\,]\, ,\, \exists xy[\, f(x) = f(y)\,]\,\}$ consistent?

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6. Does $\forall x[\, f(x) = x\,]$ follow logically from $\{\, \forall x[\, f(f(x)) = f(x)\,]\,,\, \forall x[\, g(f(x)) = x\,]\,\}$?
7. Does $\forall xy[\, x \neq y \rightarrow f(x) \neq f(y)\,]$ follow logically from $\{\, \forall x[\, g(f(x)) = x\,]\,\}$?
8. Does $\forall x \exists y[\, f(y) = x\,]$ follow logically from $\{\, \forall x[\ g(f(x)) = x\,]\,\}$?
9. Does $\forall x[\, f(g(x)) = x\,]$ follow logically from $\{\, \forall x[\ g(f(x)) = x\,]\,\}$?
10. Does $\exists x[\ g(x) = c\ ]$ follow logically from $\{\, \forall x[\, g(f(x)) = x\,]\,\}$?

PART II

Let e be a constant symbol, let S_0 and S_1 be unary function symbols, let $\circ$ be a binary function symbol and, furthermore, let $\mathcal{L}$ be the first-order language $\{e, S_0, S_1, \circ\}$ and T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(T_1) $\forall xy[\ S_0(x) \neq e\ \wedge S_1(x) \neq e\ ]$

(T_2) $\forall xy[\ x \neq y\ \rightarrow\ (S_0(x) \neq S_0(y)\ \wedge\ S_1(x) \neq S_1(y))\ ]$

(T_3) $\forall xy[\ S_0(x) \neq S_1(y)\ ]$

(T_4) $\forall x[\ e \circ x = x\ ]$

(T_5) $\forall xy[\ S_0(y) \circ x = S_0(y \circ x)\ ]$

(T_6) $\forall xy[\ S_1(y) \circ x = S_1(y \circ x)\ ]$.

We have $T \vdash S_0(e) \circ S_0(e) \neq S_1(e)$ and $T \vdash S_0(S_0(e)) \neq S_0(e)$.

Problem 4 *(weight 5 %)*

Name the non-logical axioms of T we need to deduce $S_0(e) \circ S_0(e) \neq S_1(e)$. Give a brief answer.

Problem 5 *(weight 5 %)*

Name the non-logical axioms of T we need to deduce $S_0(S_0(e)) \neq S_0(e)$. Give a brief answer.

Problem 6 *(weight 10 %)*

Show that $\{\, T_1, T_2, T_4, T_5, T_6\,\} \nvdash T_3$.

Problem 7 *(weight 10 %)*

Show that $\{\, T_1, T_2, T_4, T_5, T_6\,\} \nvdash \neg T_3$.

We define the *canonical* $\mathcal{L}$-terms inductively: e is a canonical term; $S_0(s)$ is a canonical term if s is a canonical term; $S_1(s)$ is a canonical term if s is a canonical term. (So a canonical term is a variable-free term with no occurrences of $\circ$.)

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> **Theorem I**. For every variable-free $\mathcal{L}$-term t, there exists a canonical term s such that $T \vdash t = s$.

Problem 8 *(weight 10 %)*

Prove Theorem I.

PART III

The next theorem is also known as the Compactness Theorem for first-order logic.

> **Theorem II**. Let $\mathcal{L}$ be a first-order language, and let Σ be a set of $\mathcal{L}$-formulas. If every finite subset of Σ has a model, then Σ has a model.

Problem 9 *(weight 10 %)*

Prove Theorem II. The proof should refer to the Completeness Theorem for first-order logic (do not prove the Completeness Theorem).

Let $\mathcal{L}_{NT}$ be the language of number theory, that is, the language $\{ 0, S, +, \cdot, E, < \}$. Let $\mathfrak{N}$ be the standard $\mathcal{L}_{NT}$-structure, and let $Th(\mathfrak{N})$ denote the theory of $\mathfrak{N}$, that is,

$$Th(\mathfrak{N}) \;=\; \{ \phi \mid \phi \text{ is an } \mathcal{L}_{NT}\text{-formula and } \mathfrak{N} \models \phi \} \; .$$

Let $\mathbb{Q}$ denote the set of rational numbers. For each $i \in \mathbb{Q}$ we introduce a unique constant symbol c_i. Let $\mathcal{L}_*$ be $\mathcal{L}_{NT}$ extended by $\{c_i \mid i \in \mathbb{Q}\}$. Let

$$\Sigma \;=\; Th(\mathfrak{N}) \,\cup\, \{ \; c_i < c_j \mid i, j \in \mathbb{Q} \text{ and } i < j \; \} \,\cup\, \{ \; \exists x[SSS0 + x = c_i] \mid i \in \mathbb{Q} \; \}$$

and

$$\Gamma \;=\; Th(\mathfrak{N}) \,\cup\, \{ \; c_i < c_j \mid i, j \in \mathbb{Q} \text{ and } i < j \; \} \,\cup\, \{ \; \exists x[c_i + x = SSS0] \mid i \in \mathbb{Q} \; \} \; .$$

Obviously, Σ and Γ are sets of $\mathcal{L}_*$-formulas.

Problem 10 *(weight 10 %)*

Prove that one of the two sets Σ and Γ has a model. Prove that the other set does not have have model.

Let $\mathfrak{A}$ be an $\mathcal{L}_{NT}$ structure. An infinite sequence $a_0, a_1, a_2, \ldots$ is an $\mathfrak{A}$-*predecessor chain* if $S^{\mathfrak{A}}(a_{i+1}) = a_i$ (for all $i \in \mathbb{N}$). An $\mathfrak{A}$-predecessor chain $a_0, a_1, a_2, \ldots$ *lies below* an $\mathfrak{A}$-predecessor chain $b_0, b_1, b_2, \ldots$ if $a_0 <^{\mathfrak{A}} b_i$ (for all $i \in \mathbb{N}$). If $a_0, a_1, a_2, \ldots$ lies below $b_0, b_1, b_2, \ldots$, then $b_0, b_1, b_2, \ldots$ *lies above* $a_0, a_1, a_2, \ldots$.

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Problem 11 *(weight 10 %)*

Prove that $Th(\mathfrak{N})$ has a model $\mathfrak{A}$ such that

- there are infinitely many $\mathfrak{A}$-predecessor chains
- if one $\mathfrak{A}$-predecessor lies below another $\mathfrak{A}$-predecessor chain, then there is an $\mathfrak{A}$-predecessor chain in between, that is, a chain that lies above one of the chains and below the other.

END