UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Monday, December 19, 2022.

Examination hours: 15:00 – 19:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I

Let Q be a unary relation symbol, let S be a unary function symbol, let 0 be a constant symbol and let $\mathcal{L}$ be the language $\{0, S, Q\}$. Furthermore, let

$$\Sigma_1 = \{\, \forall x[\, Q(Sx) \rightarrow Q(x)\,]\,,\, Q(S0)\,\}\ .$$

Problem 1 *(weight 10 %)*

Give a full Σ_1-deduction of $Q(0)$. Name all the logical axioms and inference rules involved in the deduction.

Problem 2 *(weight 10 %)*

Give a full Σ_1-deduction of $0 = x \rightarrow Q(x)$. Name all the logical axioms and inference rules involved in the deduction.

For any natural number n, we define the numeral $\overline{n}$ by $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$. Let

$$\Sigma_n = \{\, \forall x[\, Q(Sx) \rightarrow Q(x)\,]\,,\, Q(\overline{n})\,\}\ .$$

Problem 3 *(weight 10 %)*

Do we have $\Sigma_{17} \models \Sigma_{16}$? Do we have $\Sigma_{16} \models \Sigma_{17}$? Justify your answers.

Let $\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$.

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Problem 4 *(weight 10 %)*

Give an $\mathcal{L}$-sentence ϕ sucht that $\Sigma \nvdash \phi$ and $\Sigma \nvdash \neg\phi$. Prove that we indeed have $\Sigma \nvdash \phi$ and $\Sigma \nvdash \neg\phi$.

PART II

Let f be a binary function symbol, let c be a constant symbol and let $\mathcal{L}$ be the language $\{c, f\}$. Let

- $\Gamma_1 \ = \ \{ \ \forall x[\ x = c \ ] \, , \ f(c,c) = c \ \}$
- $\Gamma_2 \ = \ \{ \ \forall x[\ x = c \ ] \, , \ \neg(f(c,c) = c) \ \}$
- $\Gamma_3 \ = \ \{ \ \forall x[\ \neg(x = c) \ ] \, , \ \neg(f(c,c) = c) \ \}$
- $\Gamma_4 \ = \ \{ \ \neg\forall x[\ x = c \ ] \, , \ f(c,c) = c \ \}$.

Problem 5 *(weight 10 %)*

What does it mean that a set of first-order formulas is consistent? (Give the definition.) Is Γ_1 consistent? Is Γ_2 consistent? Is Γ_3 consistent? Is Γ_4 consistent? Justify your answers.

For any $\mathcal{L}$-term t, let $(t)^x_y$ denote the term t where every occurrence of the variable x is replaced by the variable y. E.g., $(ffv_1cfv_2v_1)^{v_1}_{v_3}$ denotes the term $ffv_3cfv_2v_3$ and $(fcc)^{v_3}_{v_7}$ denotes the term fcc.

Theorem 1. For any variables x, y and any $\mathcal{L}$-term t, we have

$$\vdash \ x = y \ \rightarrow \ t = (t)^x_y \ .$$

Problem 6 *(weight 10 %)*

Prove Theorem 1. Use induction on the structure of the term t.

Problem 7 *(weight 10 %)*

Do we have $\vdash t = (t)^x_y \rightarrow x = y$ (for any variables x, y and any $\mathcal{L}$-term t)? Do we have $\vdash \neg(t = (t)^x_y \rightarrow x = y)$ (for any variables x, y and any $\mathcal{L}$-term t)? Justify your answers.

PART III

Recall the language $\mathcal{L}_{NT}$, that is, the language $\{0, S, +, \cdot, E, <\}$, and its standard structure $\mathfrak{N}$. Some $\mathcal{L}_{NT}$-formulas contain bounded quantifiers, and some $\mathcal{L}_{NT}$-formulas are Δ-formulas.

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Problem 8 *(weight 5 %)*

What is a bounded quantifier? What is a Δ-formula?

Let $\phi(x) :\equiv (\exists y)[y + y = x]$, and let $\psi(x) :\equiv \neg(\exists y)[y + y = x]$.

Problem 9 *(weight 5 %)*

Give a Δ-formula $\phi_0(x)$ such that $\mathfrak{N} \models \phi_0(\overline{a})$ if and only if $\mathfrak{N} \models \phi(\overline{a})$ (for any natural number a). Give a Δ-formula $\psi_0(x)$ such that $\mathfrak{N} \models \psi_0(\overline{a})$ if and only if $\mathfrak{N} \models \psi(\overline{a})$ (for any natural number a).

Let p_i denote the i'th prime, that is, $p_1 = 2$ and $p_2 = 3$ and so on. We encode a nonempty finite sequence of natural numbers $a_1, a_2, \ldots, a_k$ as the single natural number $\langle a_1, a_2, \ldots, a_k \rangle$ where

$$\langle a_1, a_2, \ldots, a_k \rangle = p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot \ldots \cdot p_k^{a_k+1} \ .$$

The Δ-formula *IthElement*(x_1, x_2, x_3) is known from Leary & Kristiansen's textbook. We have $\mathfrak{N} \models$ *IthElement*$(\overline{b}, \overline{i}, \overline{a})$ if and only if there exists a sequence of natural numbers $a_1, a_2, \ldots, a_k$ such that $a = \langle a_1, a_2, \ldots, a_k \rangle$ and $a_i = b$.

We define the sequence $F_0, F_1, F_2, \ldots$ of Fibonacci numbers by $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

Problem 10 *(weight 10 %)*

Give an $\mathcal{L}_{NT}$-formula $\theta(x_1, x_2)$ such that $\mathfrak{N} \models \theta(\overline{i}, \overline{m})$ if and only if $F_i = m$. Use the formula *IthElement*(x_1, x_2, x_3) to construct $\theta(x_1, x_2)$.

Recall the first-order theory N from Leary & Kristiansen's book (N is given by 11 nonlogical $\mathcal{L}_{NT}$-axioms).

Problem 11 *(weight 10 %)*

Give an $\mathcal{L}_{NT}$-formula $\eta(x_1, x_2)$ such that

(1) $N \vdash \eta(\overline{i}, \overline{m})$ if $F_i = m$

(2) $N \vdash \neg\eta(\overline{i}, \overline{m})$ if $F_i \neq m$

and explain why (1) and (2) hold.

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