Part I

Theorem. Let Σ be a set of $\mathcal{L}$-formulas, and let ϕ be an $\mathcal{L}$-sentence. Then,

(i) $\Sigma \vdash \phi$ if, and only if, $\Sigma \models \phi$

(ii) Σ has a model if, and only if, Σ is consistent

(iii) if every finite subset of Σ has a model, then Σ has a model.

Problem 1

Explain briefly what it means that a set of formulas is consistent.

Problem 2

Prove that (i) and (ii) are equivalent.

Problem 3

Prove that (i) implies (iii).

Part II

Lemma (A). Let $\mathcal{L}$ be a first-order language. For any $\mathcal{L}$-theory T, any $\mathcal{L}$-terms s, t, u, we have

(i) $T \vdash s = s$

(ii) $T \vdash s = t \wedge t = u \;\rightarrow\; s = u$

(iii) $T \vdash s = t \;\rightarrow\; t = s$.

Problem 1

Prove Clause (i) of Lemma (A) by giving a T-derivation where s is an arbitrary $\mathcal{L}$-term.

———————————— Solution:

Let $\top \equiv \forall y[y = y] \vee \neg\forall y[y = y]$.

The following derivation proves $T \vdash s = s$.

1.	$x = x$	E1
2.	$\top \;\rightarrow\; x = x$	1,PC
3.	$\top \;\rightarrow\; \forall x[x = x]$	2,QR
4.	$\forall x[x = x]$	3,PC
5.	$\forall x[x = x] \;\rightarrow\; s = s$	Q1
6.	$s = s$	5,6,PC

————————————

Let $\mathcal{L}$ be the first-order language $\{\leq\}$ where $\leq$ is a binary relation symbol, and let T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(A_1) $\forall x[x \leq x]$

(A_2) $\forall xyz[x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z]$

(A_3) $\forall xy[x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y]$.

Lemma (B). $T \vdash \forall xy[x \leq y \ \wedge \ y \leq x \ \leftrightarrow \ x = y]$.

Problem 2

Prove Lemma (B) by giving a T-derivation. [Hint: use Lemma (A).]

——————————— Solution:

We will use the Lemma 2.7.2 from Leary's book:

$$\Sigma \vdash \theta \text{ if and only if } \Sigma \vdash \forall x\theta. \tag{(*)}$$

1.	$x = x \wedge x = y \ \rightarrow \ (x \leq x \ \rightarrow \ x \leq y)$	E3
2.	$x = x$	(A)
3.	$x \leq x$	A_1,(*)
4.	$x = y \ \rightarrow \ x \leq y$	1,2,3,PC
5.	$x = y \wedge x = x \ \rightarrow \ (x \leq x \ \rightarrow \ y \leq x)$	E3
6.	$x = y \ \rightarrow \ y \leq x$	2,3,5,PC
7.	$x = y \ \rightarrow \ (x \leq y \wedge y \leq x)$	4,6,PC
8.	$(x \leq y \wedge y \leq x) \ \rightarrow \ x = y$	A_3,(*)
9.	$x = y \ \leftrightarrow \ (x \leq y \wedge y \leq x)$	7,8,PC
10.	$\forall xy[x \leq y \ \wedge \ y \leq x \ \leftrightarrow \ x = y]$	9,(*)

———————————

For any terms s and t, let $s < t \equiv s \leq t \wedge s \neq t$, that is, $s < t$ is shorthand for $s \leq t \wedge s \neq t$.

Lemma (C). $T \vdash \forall xy[x < y \ \rightarrow \ \neg y \leq x]$ and $T \vdash \forall xy[\neg x \leq y \ \rightarrow \ y \neq x]$.

Problem 3

Prove Lemma (C) by giving T-derivations. [Hint: use Lemma (B).]

——————————— Solution:

The derivation below shows that the lemma holds.

1.	$\forall xy[x \le y \ \wedge\ y \le x \ \leftrightarrow\ x = y]$	(B)
2.	$x \le y \ \wedge\ y \le x \ \leftrightarrow\ x = y$	(*)
3.	$x \le y \ \wedge\ x \ne y \ \rightarrow\ \neg y \le x$	2,PC
4.	$x < y \ \rightarrow\ \neg y \le x$	3, $\equiv$
5.	$\forall xy[x < y \ \rightarrow\ \neg y \le x]$	4,(*)
6.	$\neg x \le y \ \rightarrow\ x \ne y$	2,PC
7.	$y = x \rightarrow x = y$	(A)
8.	$\neg x \le y \ \rightarrow\ y \ne x$	6,7,PC
9.	$\forall xy[\neg x \le y \ \rightarrow\ y \ne x]$	8,(*)

We extend the language $\mathcal{L}$ with a unary function symbol f, and we extend the theory T by the axioms

(A_4) $\forall x[x < f(x)]$

(A_5) $\forall xy[x < y \ \rightarrow\ f(x) < f(y)]$.

Problem 4

Prove that the axiom A_5 is independent of the other axioms; that is, prove $A_1, A_2, A_3, A_4 \not\vdash A_5$ and $A_1, A_2, A_3, A_4 \not\vdash \neg A_5$.

——————————— Solution:

Let $\mathfrak{A}$ be the following $\mathcal{L}$-structure: The universe A is the set $\mathbb{Q}^+$ of all rational numbers. Let $\le^{\mathfrak{A}}$ be the standard ordering of $\mathbb{Q}^+$. For any $q \in A$, and let

$$f^{\mathfrak{A}}(q) \ = \ \begin{cases} q+1 & \text{if } x \in \mathbb{N} \\ q + \frac{1}{q} & \text{otherwise.} \end{cases}$$

Now, $\mathfrak{A} \models \{A_1, A_2, A_3, A_4\}$ and $\mathfrak{A} \not\models A_5$. By the Soundness Theorem for first-order logic, we have $A_1, A_2, A_3, A_4 \not\vdash A_5$.

Let $\mathfrak{B}$ be the following $\mathcal{L}$-structure: The universe B is $\mathbb{N}$. Let $\le^{\mathfrak{B}}$ be the standard ordering of $\mathbb{N}$. For any $n \in A$, and let $f^{\mathfrak{B}}(n) = n + 1$. Now, $\mathfrak{B} \models \{A_1, A_2, A_3, A_4\}$ and $\mathfrak{B} \not\models \neg A_5$. By the Soundness Theorem for first-order logic, we have $A_1, A_2, A_3, A_4 \not\vdash \neg A_5$.

Let $f^0(t) \equiv t$ and $f^{n+1}(t) \equiv f(f^n(t))$.

Problem 5

Prove that we have $T \vdash t < f^{\ell}(t)$ for any $\mathcal{L}$-term t and any $\ell > 0$. [Hints: use induction on ℓ; use Lemma (C).]

——————————— Solution:

Assume $\ell = 1$ (induction start). This case is trivial as T contains the axiom A_4.

We turn to the induction step. Our induction hypothesis is $T \vdash t < f^\ell(t)$. (We will prove $T \vdash t < f^{\ell+1}(t)$.) Now, since $t < f^\ell(t) \equiv t \leq f^\ell(t) \wedge t \neq f^\ell(t)$, we have

(i) $T \vdash t \leq f^\ell(t)$

(ii) $T \vdash t \neq f^\ell(t)$

(We will prove $T \vdash t \leq f^{\ell+1}(t)$ and $T \vdash t \neq f^{\ell+1}(t)$. Then, $T \vdash t < f^{\ell+1}(t)$ because of the rule PC)

We have the following T-derivation:

1.	$f^\ell(t) < f^{\ell+1}(t)$	inst. of A_4
2.	$f^\ell(t) \leq f^{\ell+1}(t)$	1,PC
3.	$t \leq f^\ell(t)$	(i)
4.	$t \leq f^\ell(t) \ \wedge\ f^\ell(t) \leq f^{\ell+1}(t) \ \rightarrow\ t \leq f^{\ell+1}(t)$	inst. of A_2
5.	$t \leq f^{\ell+1}(t)$	

This proves $T \vdash t \leq f^{\ell+1}(t)$. The next derivation proves that $T \vdash t \neq f^{\ell+1}(t)$ holds.

1.	$f^{\ell+1}(t) \leq t \ \wedge\ t \leq f^\ell(t) \ \rightarrow\ f^{\ell+1}(t) \leq f^\ell(t)$	inst. of A_2
2.	$f^\ell < f^{\ell+1}(t)$	inst. of A_4
3.	$f^\ell < f^{\ell+1}(t) \ \rightarrow \neg f^{\ell+1}(t) \leq f^\ell$	by (C)
4.	$\neg f^{\ell+1}(t) \leq f^\ell$	2,3,PC
5.	$\neg f^{\ell+1}(t) \leq t \ \vee\ \neg t \leq f^\ell(t)$	1,4,PC
6.	$t \leq f^\ell(t)$	(i)
7.	$\neg f^{\ell+1}(t) \leq t$	5,6,PC
8.	$\neg f^{\ell+1}(t) \leq t \ \rightarrow\ f^{\ell+1}(t) \neq t$	by (C)
9.	$f^{\ell+1}(t) \neq t$	by 7,8,PC
10.	$t = f^{\ell+1}(t) \ \rightarrow\ f^{\ell+1}(t) = t$	(A)
11.	$t \neq f^{\ell+1}(t)$	9,10,PC

Problem 6

Let $\mathfrak{A}$ be any model for T. Prove that there exists an $\mathcal{L}$-structure $\mathfrak{B}$ such that (i) $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent, and (ii) there exists $b_0, b_1, b_2, \ldots$ in the universe of $\mathfrak{B}$ such that

$$f^{\mathfrak{B}}(b_{i+1}) \quad <^{\mathfrak{B}} \quad f^{\mathfrak{B}}(b_i)$$

for any $i \in \mathbb{N}$.

——————————— Solution:

Let $\mathfrak{A}$ be a model for T. We know that $T \vdash t < f^{\ell+1}(t)$ holds for any $\ell > 0$. Hence, by the Soundness Theorem, we know that $\mathfrak{A}$ contains an infinite chain

$$a_0 \;<^{\mathfrak{A}}\; a_1 \;<^{\mathfrak{A}}\; a_2 \;<^{\mathfrak{A}}\; a_3 \;<^{\mathfrak{A}}\; \dots$$

Let $\mathcal{L}'$ be $\mathcal{L}$ extended by the constants $c_0, c_1, c_2, \dots$. Let T' be the $\mathcal{L}'$-theory with get when T is extended by $\{c_{i+1} < c_i \mid i \in \mathbb{N}\}$. Any finite subset of T' has a model since $\mathfrak{A}$ contains an infinite chain. By the Compactness Theorem, T' has a model $\mathfrak{B}'$. The $\mathcal{L}'$-structure $\mathfrak{B}'$ can easily be turned into $\mathcal{L}$-structure $\mathfrak{B}$ that is elementary equivalent to $\mathfrak{A}$. Moreover, the exists elements $b_0, b_1, b_2, \dots$ in the universe of $\mathfrak{B}$ such that

$$\dots b_{i+1} \;<^{\mathfrak{B}}\; b_i \;<^{\mathfrak{B}}\; \dots \;<^{\mathfrak{B}}\; b_2 \;<^{\mathfrak{B}}\; b_1 \;<^{\mathfrak{B}}\; b_0 \;.$$

Now, as $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent, the axiom A_5 holds in $\mathfrak{B}$, that is,

$$\mathfrak{B} \models \forall xy[x < y \;\rightarrow\; f(x) < f(y)] \;.$$

Hence, we have

$$\dots f^{\mathfrak{B}}(b_{i+1}) \;<^{\mathfrak{B}}\; f^{\mathfrak{B}}(b_i) \;<^{\mathfrak{B}}\; \dots \;<^{\mathfrak{B}}\; f^{\mathfrak{B}}(b_2) \;<^{\mathfrak{B}}\; f^{\mathfrak{B}}(b_1) \;<^{\mathfrak{B}}\; f^{\mathfrak{B}}(b_0) \;.$$

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