Problem 1

1 $\forall xy[Rx \wedge Ry \to x=y] \to \forall y[Ra \wedge Ry \to a=y]$ (Q1)

2. $\forall y[Ra \wedge Ry \to a=y] \to (Ra \wedge Rb \to a=b)$ (Q1)

3 $\forall xy[Rx \wedge Ry \to x=y]$ Σ_1

4 $Ra \wedge Rb \to a=b$ 1, 2, 3, (PC)

5 Ra Σ_1

6 Rb Σ_1

7 $a=b$ 4, 5, 6, (PC)

Problem 2

The universe A is given by

$$A = \{0, 1, 2\}.$$

$$a^{\mathfrak{A}} = 0$$

$$b^{\mathfrak{A}} = 0$$

$$R^{\mathfrak{A}} = \{0\}.$$

Problem 3

1. $\forall xy[Rx \wedge Ry \rightarrow x=y] \rightarrow \forall y[Ra \wedge Ry \rightarrow a=y]$ (Q1)
2. $\forall y[Ra \wedge Ry \rightarrow x=y] \rightarrow (Ra \wedge Rb \rightarrow a=b)$ (Q1)
3. $\forall xy[Rx \wedge Ry \rightarrow x=y]$ Σ_2
4. $\neg a=b$ Σ_2
5. $\neg Ra \vee \neg Rb$ 1,2,3,4, (PC).

Problem 4

The universe $\mathfrak{B}$ is given by

$$B = \{0, 1, 2\}.$$

$$a^{\mathfrak{B}} = 0$$

$$b^{\mathfrak{B}} = 1$$

$$R^{\mathfrak{A}} = \{0\}.$$

Problem 5

The set $\Sigma_1 \cup \Sigma_2$ is not consistent.

We have $\Sigma_1 \vdash a = b$ (Problem 1)

We have $\Sigma_2 \vdash \neg a = b$ (obvious)

Thus, $\Sigma_1 \cup \Sigma_2 \vdash \bot$ and $\Sigma_1 \cup \Sigma_2$ is inconsistent.

(The definition of "consistent":

A set Σ of $\mathcal{L}$-formulas is consistent

$$\Longleftrightarrow_{def}$$

$$\Sigma \nvdash \bot.$$)

Problem 6

The set $\Sigma_1 \cup \Sigma_2$ does not have a model by Problem 5 and the Soundness Theorem.

(The Soundness Theorem:

$$\Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi.$$

This is equivalent to

Σ has a model $\Rightarrow$ Σ is consistent.)

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Prolem 7

$$\exists z_1 z_2 [\, \varphi^x_{z_1} \wedge \varphi^x_{z_2} \wedge z_1 \neq z_2 \,] \wedge$$

$$\forall z_1 z_2 z_3 \left[\varphi^x_{z_1} \wedge \varphi^x_{z_2} \wedge \varphi^x_{z_3} \rightarrow (z_1 = z_2 \vee z_1 = z_3 \vee z_2 = z_3) \right]$$

Alternative answer:

$$\exists z_1 z_2 \left[\varphi^x_{z_1} \wedge \varphi^x_{z_2} \wedge z_1 \neq z_2 \wedge \forall y [\, \varphi^x_y \rightarrow (y = z_1 \vee y = z_2)] \right]$$

Problem 8

We define the structure $\mathfrak{A}$.

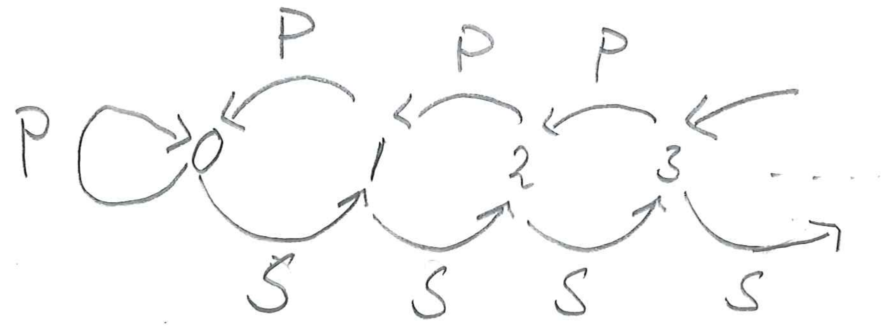
The universe A is the set of natural numbers, that is, $A = \mathbb{N}$.

$$0^{\mathfrak{A}} = 0$$

$$p^{\mathfrak{A}}(x) = \begin{cases} x-1 & \text{if } x \in \mathbb{N} \setminus \{0\} \\ 0 & \text{if } x = 0. \end{cases}$$

$$s^{\mathfrak{A}}(x) = x+1 \quad (\text{for all } x \in \mathbb{N}).$$

Figure:



Problem 9

By T_4 and logical axioms, we have

$$T \vdash PS0 = 0. \qquad (1)$$

By T_5 and logical axioms, we have

$$T \vdash SP0 \neq P0. \qquad (2)$$

By T_1, (2) and logical axioms (equality axioms must be involved), we have

$$T \vdash SP0 \neq 0 \qquad (3)$$

By (1), (3) and logical axioms, we have

$$T \vdash SP0 \neq PS0.$$

Thus, the non-logical axioms in the deduction are T_4, T_5 and T_1.

Problem 10

Let $\mathfrak{A}$ be the structure from Problem 8. We have

$$\mathfrak{A} \models \{T_1, T_2, T_3, T_4\} \text{ and } \mathfrak{A} \not\models \neg T_5.$$

By the Soundness Theorem, we have

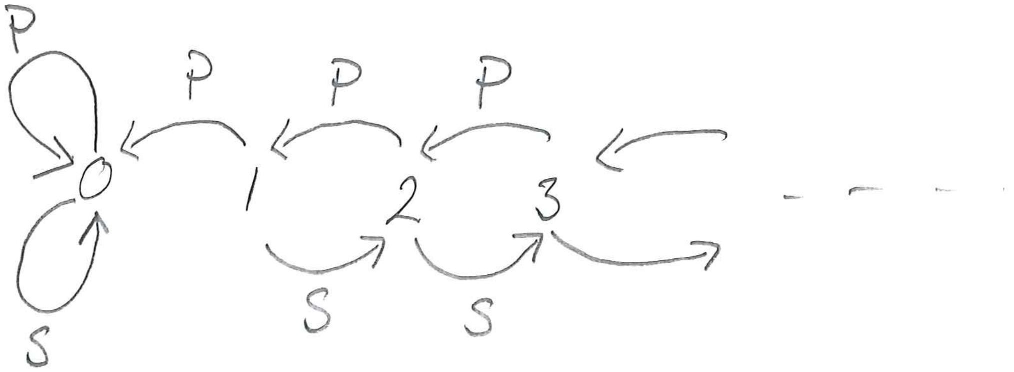
$$\{T_1, T_2, T_3, T_5\} \not\vdash \neg T_5.$$

Let $\mathfrak{B}$ be the structure given by

- the universe is $\mathbb{N}$
- $0^{\mathfrak{B}} = 0$
- $P^{\mathfrak{B}}(x) = \begin{cases} x-1 & \text{if } x \in \mathbb{N} \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$
- $S^{\mathfrak{B}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \setminus \{0\} \\ 0 & \text{if } x = 0. \end{cases}$

Problem 10 (continues)

Figure:



We have

$$\mathfrak{B} \models \{T_1, T_2, T_3, T_4\} \text{ and } \mathfrak{B} \not\models T_5$$

By the Soundness Theorem, we have

$$\{T_1, T_2, T_3, T_4\} \nvdash T_5.$$

Problem 11

Let $\mathfrak{A}$ be the structure from Problem 8.
By the Soundness Theorem, we have

$$\{T_2, T_3, T_4, T_5\} \nvdash \neg T_1$$

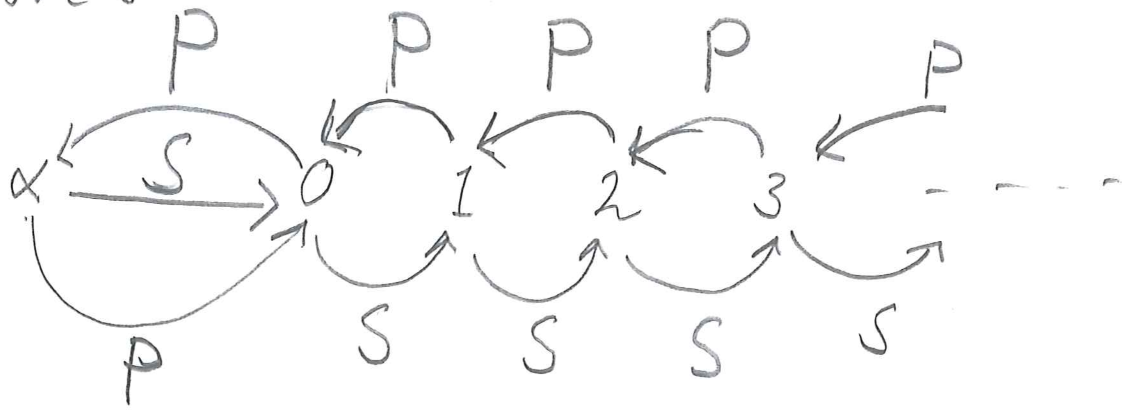
(see the soultion of Problem 10).

Let $\mathfrak{B}$ be the structure given by

- the universe is $\mathbb{N} \cup \{\alpha\}$ (and α is not a natural number)
- $0^{\mathfrak{B}} = 0$
- $$p^{\mathfrak{B}}(x) = \begin{cases} x-1 & \text{if } x \in \mathbb{N} \setminus \{0\} \\ \alpha & \text{if } x = 0 \\ 0 & \text{if } x = \alpha \end{cases}$$
- $$S^{\mathfrak{B}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ 0 & \text{if } x = \alpha \end{cases}$$

Problem 11 (continues)

Figure:



We have

$$\mathfrak{B} \models \{T_2, T_3, T_4, T_5\} \text{ and } \mathfrak{B} \not\models T_1$$

By the Soundness Theorem, we have

$$\{T_2, T_3, T_4, T_5\} \not\vdash T_1 .$$

Problem 12

Let $\mathfrak{A} \models T$. Let A denote the universe of $\mathfrak{A}$.

Assume A is finite, that is, assume

$$A = \{a_0, a_1, a_2, \ldots, a_n\}$$

for some $n \in \mathbb{N}$. Let $0^{\mathfrak{A}} = a_0$.

By T_1, we have $P^{\mathfrak{A}}(a_0) = a_0$.

By T_2 and T_3, we have

"for every $b \in \{a_0, a_1, \ldots, a_n\}$ there exists $c \in \{a_1, \ldots, a_n\}$ such that $P^{\mathfrak{A}}(c) = b$"

This is a contradiction since the set $\{a_0, a_1, \ldots, a_n\}$ contains one more element than the set $\{a_1, \ldots, a_n\}$.

Thus, we conclude that A is infinite.

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Problem 13

Use $\varphi :\equiv x=x$.

By the conjecture, we have

$$\mathfrak{A} \models (\exists^{\infty} x)[x=x] \text{ if and only if}$$

the universe of $\mathfrak{A}$ is infinite.

The formula $(\exists^{\infty} x)[x=x]$ does not exist by Exercise 8 at page 93 of "a friendly introduction to mathematical logic."

See the solution of the exercise for more detials.