UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Friday, December 15, 2017.

Examination hours: 9:00 – 13:00.

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let R and S be a unary relation symbols, and let a be a constant symbol. Let $\mathcal{L}$ be the language $\{a, R, S\}$.

Problem 1

State the Soundness Theorem for first-order logic. State the Completeness Theorem for first-order logic.

Problem 2

Let ϕ be an $\mathcal{L}$-formula such that $\nvdash \phi$. Explain why there exists an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models \phi$. Give a brief answer.

Problem 3

Below you will find six $\mathcal{L}$-formulas $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$. For each formula ϕ_i, we either have $\vdash \phi_i$ or $\nvdash \phi_i$. If $\vdash \phi_i$, you should give a detailed deduction of ϕ_i (name all the axioms and inference rules involved in the deduction). If $\nvdash \phi_i$, you should give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models \phi_i$.

- $\phi_1 :\equiv Ra \to (Sa \to Ra)$
- $\phi_2 :\equiv \exists x[Rx] \to \exists x[Sx \to Rx]$
- $\phi_3 :\equiv \forall x[Rx] \to \forall x[Sx \to Rx]$
- $\phi_4 :\equiv \exists x[Sx \to Rx] \to \exists x[Rx]$
- $\phi_5 :\equiv \forall x[Sx \to Rx] \to \forall x[Rx]$

(Continued on page 2.)

- $\phi_6 :\equiv (\forall x[Rx] \to Ra) \ \vee \ (\forall x[Sx \to Rx] \to \forall x[Rx])$

SOLUTION ———————————————— PROBLEM 3

We have $\vdash \phi_1$. Deduction:

$$1. \quad Ra \to (Sa \to Ra) \qquad \text{(PC)}$$

Yes, that is it. One formula.

We have $\vdash \phi_2$. Deduction:

1.	$Rx \to (Sx \to Rx)$	(PC)
2.	$(Sx \to Rx) \ \to \ \exists x[Sx \to Rx]$	(Q2)
3.	$Rx \ \to \ \exists x[Sx \to Rx]$	(PC), 1, 2
4.	$\exists x[Rx] \ \to \ \exists x[Sx \to Rx]$	(QR), 3

We have $\vdash \phi_3$. Deduction:

1.	$\forall x[Rx] \to Rx$	(Q1)
2.	$Rx \to (Sx \to Rx)$	(PC)
3.	$\forall x[Rx] \to (Sx \to Rx)$	(PC), 1, 2
4.	$\forall x[Rx] \to \forall x[Sx \to Rx]$	(QR), 3

We have $\not\vdash \phi_4$. We give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models \phi_4$: The universe is $\{\bullet\}$ (any nonempty set will work). Let $R^{\mathfrak{A}} = S^{\mathfrak{A}} = \emptyset$. *Explanation:* We have $\mathfrak{A} \not\models \exists x[Sx]$. Thus, $\mathfrak{A} \models \exists x[Sx \to Rx]$. Furthermore, we have $\mathfrak{A} \not\models \exists x[Rx]$. Thus, $\mathfrak{A} \not\models \exists x[Sx \to Rx] \to \exists x[Rx]$. (We are not asked to justify our answer. The explanation is superfluous.)

We have $\not\vdash \phi_5$. We give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models \phi_5$: The universe is $\{0, 1, 2\}$. Let $S^{\mathfrak{A}} = \{0\}$. Let $R^{\mathfrak{A}} = \{0, 1\}$.

We have $\vdash \phi_6$. Deduction:

1.	$\forall x[Rx] \to Ra$	(Q1)
2.	$(\forall x[Rx] \to Ra) \ \vee \ (\forall x[Sx \to Rx] \to \forall x[Rx])$	(PC), 1

———————————————— END OF SOLUTION

Part II

Let $<$ be a binary relation symbol, let S be unary function symbols, and let 0 be a constant symbol. Let $\mathcal{L}$ be the language $\{0, S, <\}$. Let T be the $\mathcal{L}$-theory where we have the following non-logical axioms:

(T_1) $\forall x[\neg Sx = 0]$

(T_2) $\forall xy[Sx = Sy \to x = y]$

(T_3) $\forall x[\neg Sx = x]$

(T_4) $\forall x[\neg x < 0]$

(T_5) $\forall xy[x < Sy \ \leftrightarrow \ (x < y \vee x = y)]$

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Problem 4

Prove that the axiom T_3 is independent of the other axioms of T, that is, prove that

$$\{T_1, T_2, T_4, T_5\} \not\vdash T_3 \quad \text{and} \quad \{T_1, T_2, T_4, T_5\} \not\vdash \neg T_3 \ .$$

SOLUTION ———————————————— PROBLEM 4

We give an $\mathcal{L}$-structure $\mathfrak{N}$ such that $\mathfrak{N} \models \{T_1, T_2, T_4, T_5\}$ and $\mathfrak{N} \not\models \neg T_3$: The universe is $\mathbb{N}$ (the set of natural numbers). Let $<^{\mathfrak{N}}$ be the standard strict ordering of the natural numbers, i.e.

$$<^{\mathfrak{N}} \ = \ \{ \, (a, b) \mid a, b \in \mathbb{N} \text{ and } a < b \, \} \ .$$

Let $S^{\mathfrak{N}}$ be the successor function, i.e. $S^{\mathfrak{N}}(x) = x + 1$, and let $0^{\mathfrak{N}} = 0$. It is easy to see that we have $\mathfrak{N} \models \{T_1, T_2, T_4, T_5\}$ and $\mathfrak{N} \not\models \neg T_3$. By the Soundness Theorem for first-order logic, we have $\{T_1, T_2, T_4, T_5\} \not\vdash \neg T_3$.

We give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \{T_1, T_2, T_4, T_5\}$ and $\mathfrak{A} \not\models T_3$: The universe is $\mathbb{N} \cup \{\omega\}$ (where ω of course if something else than a natural number). Let

$$<^{\mathfrak{A}} \ = \ \{ \, (a, b) \mid a, b \in \mathbb{N} \text{ and } a < b \, \} \ \cup \ \{ \, (\omega, \omega) \, \} \ .$$

Let

$$S^{\mathfrak{A}}(a) = \begin{cases} a + 1 & \text{if } a \in \mathbb{N} \\ a & \text{if } a = \omega \end{cases}$$

and let $0^{\mathfrak{N}} = 0$. It is not hard to check that we have $\mathfrak{A} \models \{T_1, T_2, T_4, T_5\}$ and $\mathfrak{A} \not\models T_3$. By the Soundness Theorem for first-order logic, we have $\{T_1, T_2, T_4, T_5\} \not\vdash T_3$.

———————————————— END OF SOLUTION

Let $\phi(x)$ be any $\mathcal{L}$-formula (as usual $\phi(t)$ denotes $\phi(x)$ where every free occurrence of the variable x is replaced by the term t). We will consider three axiom schemes.

The scheme of Zero Intolerance. This is the scheme

$$\forall x[\phi(Sx)] \rightarrow \forall x[\phi(x)] \qquad (Z)$$

The theory T_Z is the theory T extended by this axiom scheme.

The scheme of Pseudo Induction. This is the scheme

$$(\, \phi(0) \ \wedge \ \forall x[\phi(Sx)] \,) \ \rightarrow \ \forall x[\phi(x)] \qquad (P)$$

The theory T_P is the theory T extended by this axiom scheme.

The scheme of Induction. This is the scheme

$$(\, \phi(0) \ \wedge \ \forall x[\phi(x) \rightarrow \phi(Sx)] \,) \ \rightarrow \ \forall x[\phi(x)] \qquad (I)$$

The theory T_I is the theory T extended by this axiom scheme.

(Continued on page 4.)

Problem 5

Give a T_Z-deduction of $\neg 0 = 0$. Give a full deduction. Name all the axioms and inference rules involved in the deduction.

SOLUTION ———————————————— PROBLEM 5

1.	$\forall x[\neg Sx = 0] \ \rightarrow \ \forall x[\neg x = 0]$	(Z)
2.	$\forall x[\neg Sx = 0]$	(T_1)
3.	$\forall x[\neg x = 0]$	1, 2, (PC)
4.	$\forall x[\neg x = 0] \ \rightarrow \ \neg 0 = 0$	$(Q1)$
5.	$\neg 0 = 0$	3, 4, (PC)

———————————————— END OF SOLUTION

Problem 6

Explain why we have $T_Z \vdash \theta$ for every $\mathcal{L}$-formula θ.

SOLUTION ———————————————— PROBLEM 6

We have $T_Z \vdash \neg 0 = 0$ and $T_Z \vdash 0 = 0$. For any θ, we have $T_Z \vdash \theta$ as θ follows tautologically from $\neg 0 = 0$ and $0 = 0$ (the theory T_Z is inconsistent).

———————————————— END OF SOLUTION

Problem 7

Prove that

$$T_P \vdash \forall x[\, x = 0 \ \vee \ \exists y[Sy = x]\,] \, .$$

Sketch a T_P-deduction of the formula. Name all the non-logical axioms involved in the deduction.

SOLUTION ———————————————— PROBLEM 7

We will use the scheme of Pseudo Induction with $\phi(x) :\equiv x = 0 \vee \exists y[Sy = x]$.

By $(E1)$ and other logical axioms, we have $T_P \vdash 0 = 0$. Thus, by (PC)

$$T_P \vdash 0 = 0 \ \vee \ \exists y[Sy = 0] \, .$$

This shows that $T_P \vdash \phi(0)$.

By logical axioms, we have $\exists y[Sy = Sx]$. Thus, by (PC)

$$T_P \vdash Sx = 0 \ \vee \ \exists y[Sy = Sx] \, .$$

This shows that $T_P \vdash \phi(Sx)$. Furthermore, by $(Q1)$ and other logical axioms, we have $T_P \vdash \forall x[\phi(Sx)]$.

Thus, we have $T_P \vdash \phi(0)$ and $T_P \vdash \forall x[\phi(Sx)]$. By (PC) and (P), we have $T_P \vdash \forall x[\phi(x)]$, that is

$$T_P \vdash \forall x[\, x = 0 \ \vee \ \exists y[Sy = x]\,] \, .$$

———————————————— END OF SOLUTION

(Continued on page 5.)

Problem 8

Prove that

$$T \not\vdash \forall x[\, x = 0 \ \vee \ \exists y[Sy = x]\,] \ .$$

SOLUTION ———————————————— PROBLEM 8

We will give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$ and

$$\mathfrak{A} \not\models \forall x[\, x = 0 \ \vee \ \exists y[Sy = x]\,] \ .$$

First we give the universe A. Let B be an countably infinite set containing the elements $\beta_0, \beta_1, \beta_2 \ldots$ (for each i, we have $\beta_i \not\in \mathbb{N}$). Let $A = \mathbb{N} \cup B$.

Let $0^{\mathfrak{A}} = 0$, let

$$S^{\mathfrak{A}}(a) = \begin{cases} a+1 & \text{if } a \in \mathbb{N} \\ \beta_{i+1} & \text{if } a = \beta_i \end{cases}$$

and let

$$<^{\mathfrak{A}} \ = \ \{\, (a,b) \mid a,b \in \mathbb{N} \text{ and } a < b \,\} \ \cup \ \{\, (\beta_i, \beta_j) \mid i < j \,\} \ .$$

It is obvious that $\mathfrak{A} \models \{T_1, T_2, T_3, T_4\}$. We argue that $\mathfrak{A} \models T_5$: We have

$$a <^{\mathfrak{A}} S^{\mathfrak{A}}(b) \ \leftrightarrow \ (a <^{\mathfrak{A}} b \vee a = b) \qquad (*)$$

when $a, b \in \mathbb{N}$ (as $<^{\mathfrak{A}}$ restricted to $\mathbb{N}$ is the standard strict ordering of $\mathbb{N}$). Furthermore, (*) holds when $a, b \in B$. If $a \in \mathbb{N}$ and $b \in B$, (*) holds since both sides of the bi-implication is false. If $b \in \mathbb{N}$ and $a \in B$, (*) holds since both sides of the bi-implication is false. Thus, we conclude that $\mathfrak{A} \models T$.

We have

$$\mathfrak{A} \not\models \forall x[\, x = 0 \ \vee \ \exists y[Sy = x]\,]$$

since $\beta_0 \neq 0^{\mathfrak{A}}$ and there is no y in the universe such that $S^{\mathfrak{A}}(y) = \beta_0$.

———————————————— END OF SOLUTION

Problem 9

Let θ be any $\mathcal{L}$-formula. Prove that $T_P \vdash \theta$ implies $T_I \vdash \theta$.

SOLUTION ———————————————— PROBLEM 9

Let $\phi(x)$ be an arbitrary $\mathcal{L}$-formula. We have (see Problem 3)

$$\vdash \forall x[\phi(Sx)] \ \to \ \forall x[\phi(x) \to \phi(Sx)] \ .$$

Thus, we also have

$$T_I \vdash \forall x[\phi(Sx)] \ \to \ \forall x[\phi(x) \to \phi(Sx)] \ . \qquad ((*))$$

It is trivial that

$$T_I \vdash (\, \phi(0) \ \wedge \ \forall x[\phi(x) \to \phi(Sx)]\,) \ \to \ \forall x[\phi(x)] \ . \qquad (**)$$

By (*), (**) and (PC), we have

$$T_I \vdash (\, \phi(0) \ \wedge \ \forall x[\phi(Sx)]\,) \ \to \ \forall x[\phi(x)] \ .$$

This shows that the scheme of Pseudo Induction (P) is available in the theory T_I. Hence, any formula deducible from the axioms of T_P will also be deducible from the axioms of T_I.

———————————————— END OF SOLUTION

(Continued on page 6.)

Problem 10

Does it exist an $\mathcal{L}$-formula η such that $T_I \vdash \eta$ and $T_P \nvdash \eta$? Justify your answer. If you can, prove that your answer is correct.

SOLUTION ———————————————— PROBLEM 10

We have $T_I \vdash \forall x[\neg SSx = x]$ and $T_P \nvdash \forall x[\neg SSx = x]$.

First we argue that $T_I \vdash \forall x[\neg SSx = x]$. We have

$$T_I \vdash [\neg SS0 = 0] \qquad (*)$$

by (T_1). We have

$$T_I \vdash \forall x[\neg SSx = x \quad \rightarrow \quad \neg SSSx = Sx] \qquad (**)$$

by (T_2). By (*), (**), (I) and (PC), we have $T_I \vdash \forall x[\neg SSx = x]$ (we use the scheme of Induction with $\phi(x) :\equiv \neg SSx = x$).

We give an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T_P$ and $\mathfrak{A} \not\models \forall x[\neg SSx = x]$. The universe of $\mathfrak{A}$ is $\mathbb{N} \cup \{\alpha, \beta\}$ (where $\alpha, \beta \notin \mathbb{N}$). Let $0^{\mathfrak{A}} = 0$, let $S^{\mathfrak{A}}(a) = a + 1$ when $a \in \mathbb{N}$, let $S^{\mathfrak{A}}(\alpha) = \beta$, let $S^{\mathfrak{A}}(\beta) = \alpha$. Let

$$<^{\mathfrak{A}} \;=\; \{\, (a, b) \mid a, b \in \mathbb{N} \text{ and } a < b \,\} \;\cup\; \{\, (\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta) \,\} \;.$$

We have $S^{\mathfrak{A}}S^{\mathfrak{A}}(\alpha) = \alpha$. Thus $\mathfrak{A} \not\models \forall x[\neg SSx = x]$. It is possible to check that $\mathfrak{A} \models T_P$ (It is obvious that $\mathfrak{A} \models \{T_1, T_2, T_4\}$. Some work is required to check that $\mathfrak{A} \models T_5$. Since any element in the universe either equals $0^{\mathfrak{A}}$ or is the the successor of something, $\mathfrak{A}$ satisfies the scheme of Pseudo Induction.)

———————————————— END OF SOLUTION

END