UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Tuesday, December 1, 2020.

Examination hours: 15:00 – 19:00.

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The weights might be adjusted.

PART I

Let P and Q be unary relation symbols. Let a, b and c be constant symbols. Let f and g be unary function symbols. Furthermore, x and y denote variables.

Problem 1 *(weight 10 %)*

Let $\Sigma = \{\forall x[Px]\}$. Give a full Σ-deduction of $\forall x[Qc \to Px]$. Name all the logical axioms and inference rules involved in the deduction.

—————————— Solution:

1.	$\forall x[Px] \to Px$	(Q1)
2.	$\forall x[Px]$	Σ
3.	Px	1, 2, (PC)
4.	$Qc \to Px$	3, (PC)
5.	$\top \to (Qc \to Px)$	4, (PC)
6.	$\top \to \forall x[Qc \to Px]$	5, (QR)
7.	$\forall x[Qc \to Px]$	6, (PC)

Problem 2 *(weight 10 %)*

Let $\mathcal{L}$ be the language $\{P, Q, c\}$ Give an $\mathcal{L}$ structure $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x[Qc \to Px]$ and $\mathfrak{A} \not\models \forall x[Px]$. Explain briefly why we have $\{\forall x[Qc \to Px]\} \not\vdash \forall x[Px]$.

—————————— Solution:

(Continued on page 2.)

Let the universe A be any nonempty set, e.g., let $A = \mathbb{N}$. Let $Q^{\mathfrak{A}} = \emptyset$ and $P^{\mathfrak{A}} = \emptyset$. Let $c^{\alpha} = 0$.

We have a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x[\,Qc \rightarrow Px\,]$ and $\mathfrak{A} \not\models \forall x[\,Px\,]$. Thus, $\{\,\forall x[\,Qc \rightarrow Px\,]\,\} \not\vdash \forall x[\,Px\,]$ follows by the Soundness Theorem.

Problem 3 *(weight 10 %)*

Ten Questions: Answer each question with a YES or a NO (and nothing else). If you do not answer a question, your answer to that question will be considered as wrong.

1. Is the set $\{\,a \neq b\,,\ f(a) = f(b)\,]\,\}$ consistent? **YES**
2. Is the set $\{\,a = b\,,\ f(a) \neq f(b)\,]\,\}$ consistent? **NO**
3. Is the set $\{\,f(a) = g(b)\,,\ g(b) = f(c)\,,\ g(f(a)) \neq g(f(c))\,]\,\}$ consistent? **NO**
4. Is the set $\{\,\forall xy[\,f(x) \neq f(y)\,]\,,\ \exists xy[\,f(x) \neq f(y)\,]\,\}$ consistent? **NO**
5. Is the set $\{\,\forall xy[\,f(x) = f(y)\,]\,,\ \exists xy[\,f(x) = f(y)\,]\,\}$ consistent? **YES**
6. Does $\forall x[\,f(x) = x\,]$ follow logically from $\{\,\forall x[\,f(f(x)) = f(x)\,]\,,\ \forall x[\,g(f(x)) = x\,]\,\}$? **YES**
7. Does $\forall xy[\,x \neq y \rightarrow f(x) \neq f(y)\,]$ follow logically from $\{\,\forall x[\,g(f(x)) = x\,]\,\}$? **YES**
8. Does $\forall x \exists y[\,f(y) = x\,]$ follow logically from $\{\,\forall x[\,g(f(x)) = x\,]\,\}$? **NO**
9. Does $\forall x[\,f(g(x)) = x\,]$ follow logically from $\{\,\forall x[\,g(f(x)) = x\,]\,\}$? **NO**
10. Does $\exists x[\,g(x) = c\,]$ follow logically from $\{\,\forall x[\,g(f(x)) = x\,]\,\}$? **YES**

PART II

Let e be a constant symbol, let S_0 and S_1 be unary function symbols, let $\circ$ be a binary function symbol and, furthermore, let $\mathcal{L}$ be the first-order language $\{e, S_0, S_1, \circ\}$ and T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(T_1) $\forall xy[\,S_0(x) \neq e\ \wedge S_1(x) \neq e\,]$

(T_2) $\forall xy[\,x \neq y\ \rightarrow\ (S_0(x) \neq S_0(y)\ \wedge\ S_1(x) \neq S_1(y))\,]$

(T_3) $\forall xy[\,S_0(x) \neq S_1(y)\,]$

(T_4) $\forall x[\,e \circ x = x\,]$

(T_5) $\forall xy[\,S_0(y) \circ x = S_0(y \circ x)\,]$

(T_6) $\forall xy[\,S_1(y) \circ x = S_1(y \circ x)\,]$.

We have $T \vdash S_0(e) \circ S_0(e) \neq S_1(e)$ and $T \vdash S_0(S_0(e)) \neq S_0(e)$.

(Continued on page 3.)

Problem 4 *(weight 5 %)*

Name the non-logical axioms of T we need to deduce $S_0(e) \circ S_0(e) \neq S_1(e)$. Give a brief answer.

——————————————— Solution:

We need T_3 and T_5.

Problem 5 *(weight 5 %)*

Name the non-logical axioms of T we need to deduce $S_0(S_0(e)) \neq S_0(e)$. Give a brief answer.

——————————————— Solution:

We need T_1 and T_2.

Problem 6 *(weight 10 %)*

Show that $\{T_1, T_2, T_4, T_5, T_6\} \nvdash T_3$.

——————————————— Solution:

Let $\mathfrak{A}$ be a structure where the universe is the set of natural numbers $\mathbb{N}$. Furthermore, let $e^{\mathfrak{A}} = 0$, let $S_0^{\mathfrak{A}}(x) = S_1^{\mathfrak{A}}(x) = x + 1$, and let $x^\circ y = x + y$ (standard addition). Then we have $\mathfrak{A} \models \{T_1, T_2, T_4, T_5, T_6\}$ and $\mathfrak{A} \not\models T_3$. By the Soundness Theorem, we have $\{T_1, T_2, T_4, T_5, T_6\} \nvdash T_3$.

Problem 7 *(weight 10 %)*

Show that $\{T_1, T_2, T_4, T_5, T_6\} \nvdash \neg T_3$.

——————————————— Solution:

Let $\mathfrak{B}$ be a structure where the universe is the set $\{0,1\}^*$ of all bit strings over the alphabet $\{0,1\}$ (the set includes the empty string ε). Let $e^{\mathfrak{B}} = \varepsilon$. For any $b \in \{0,1\}^*$, let $S_0^{\mathfrak{B}}(b) = 0b$ and $S_1^{\mathfrak{B}}(b) = 1b$. Then we have $\mathfrak{B} \models \{T_1, T_2, T_4, T_5, T_6\}$ and $\mathfrak{B} \not\models \neg T_3$. By the Soundness Theorem, we have $\{T_1, T_2, T_4, T_5, T_6\} \nvdash \neg T_3$.

We define the *canonical* $\mathcal{L}$-terms inductively: e is a canonical term; $S_0(s)$ is a canonical term if s is a canonical term; $S_1(s)$ is a canonical term if s is a canonical term. (So a canonical term is a variable-free term with no occurrences of $\circ$.)

> **Theorem I**. For every variable-free $\mathcal{L}$-term t, there exists a canonical term s such that $T \vdash t = s$.

Problem 8 *(weight 10 %)*

Prove Theorem I.

——————————————— Solution:

(Continued on page 4.)

> **(Claim).** For any canonical $\mathcal{L}$-terms t and s, there exists a canonical term u such that $T \vdash t \circ s = u$.

We prove the claim by induction on the structure of the canonical term t. The proof splits into the cases (i) $t :\equiv e$, (ii) $t :\equiv S_0(t')$ and (iii) $t :\equiv S_1(t')$.

(i) $t :\equiv e$. Let s be any canonical term. By T_1 we have $T \vdash e \circ s = s$. Thus the claim holds in case (i).

(ii) $t :\equiv S_0 t'$. Let s be any canonical term. By the ind. hyp. we have a canonical term u' such that

$$T \vdash t' \circ s = u' \ . \tag{A}$$

By (E2) and other logical axioms we have

$$T \vdash t' \circ s = u' \ \rightarrow S_0(t' \circ s) = S_0(u') \ . \tag{B}$$

By (A), (B) and (PC), we have

$$T \vdash S_0(t' \circ s) = S_0(u') \ . \tag{C}$$

By T_5, we have

$$T \vdash S_0(t') \circ s = S_0(t' \circ x) \ . \tag{D}$$

By (C), (D) and logical axioms, we have $T \vdash S_0(t') \circ s = S_0 u'$. Thus, the claim holds as $S_0 u'$ is a canonical term. This completes the proof of case (ii).

Case (iii) is symmetric to case (ii). Use the axiom T_6 in place of T_5.

We turn to the roof of Theorem I. We prove the theorem by induction on the structure of t. We have the cases

- $t :\equiv e$
- $t :\equiv S_0(t')$
- $t :\equiv S_1(t')$
- $t :\equiv t' \circ t''$.

Case $t :\equiv e$. By (E1) and other logical axioms, we have $T \vdash e = e$. We observe that e is a canonical term. Thus the theorem holds when $t :\equiv e$.

Case $t :\equiv S_0(t')$. By the ind. hyp. we have a canonical term s' such that $T \vdash t' = s'$. By (E2) and other logical axioms, we have $T \vdash t' = s' \rightarrow S_0(t') = S_0(s')$. By (PC), we have $T \vdash S_0(t') = S_0(s')$. Now, $S_0(s')$ is a canonical term. This completes the proof of the case $t :\equiv S_0(t')$.

Case $t :\equiv S_1(t')$. This case is symmetric to the preceding case.

(Continued on page 5.)

Case $t :\equiv t' \circ t''$. The ind. hyp. yields canonical terms s' and s'' such that $T \vdash t' = s'$ and $T \vdash t'' = s''$. By (E2) and other logical axioms, we have

$$t' = s' \wedge t'' = s'' \quad \to t' \circ t'' = s' \circ s'' \ .$$

Thus, by (PC), we have

$$T \vdash t' \circ t'' = s' \circ s'' \ . \tag{E}$$

By (Claim), we have a canonical term s such that

$$T \vdash s' \circ s'' = s \ . \tag{F}$$

By (E), (F) and logical axioms, we have $T \vdash t' \circ t'' = s$ This proves that the theorem holds when $t :\equiv t' \circ t''$.

PART III

The next theorem is also known as the Compactness Theorem for first-order logic.

> **Theorem II**. Let $\mathcal{L}$ be a first-order language, and let Σ be a set of $\mathcal{L}$-formulas. If every finite subset of Σ has a model, then Σ has a model.

Problem 9 *(weight 10 %)*

Prove Theorem II. The proof should refer to the Completeness Theorem for first-order logic (do not prove the Completeness Theorem).

Let $\mathcal{L}_{NT}$ be the language of number theory, that is, the language $\{ 0, S, +, \cdot, E, < \}$. Let $\mathfrak{N}$ be the standard $\mathcal{L}_{NT}$-structure, and let $Th(\mathfrak{N})$ denote the theory of $\mathfrak{N}$, that is,

$$Th(\mathfrak{N}) \ = \ \{ \ \phi \mid \phi \text{ is an } \mathcal{L}_{NT}\text{-formula and } \mathfrak{N} \models \phi \ \} \ .$$

Let $\mathbb{Q}$ denote the set of rational numbers. For each $i \in \mathbb{Q}$ we introduce a unique constant symbol c_i. Let $\mathcal{L}_*$ be $\mathcal{L}_{NT}$ extended by $\{c_i \mid i \in \mathbb{Q}\}$. Let

$$\Sigma \ = \ Th(\mathfrak{N}) \ \cup \ \{ \ c_i < c_j \mid i,j \in \mathbb{Q} \text{ and } i < j \ \} \ \cup \ \{ \ \exists x[SSS0 + x = c_i] \mid i \in \mathbb{Q} \ \}$$

and

$$\Gamma \ = \ Th(\mathfrak{N}) \ \cup \ \{ \ c_i < c_j \mid i,j \in \mathbb{Q} \text{ and } i < j \ \} \ \cup \ \{ \ \exists x[c_i + x = SSS0] \mid i \in \mathbb{Q} \ \} \ .$$

Obviously, Σ and Γ are sets of $\mathcal{L}_*$-formulas.

Problem 10 *(weight 10 %)*

Prove that one of the two sets Σ and Γ has a model. Prove that the other set does not have have model.

—————————— Solution:

The set Σ has a model.

(Continued on page 6.)

Proof (sketch). Argue that any finite subset of Σ has model. Use Theorem II (the Compactness Theorem) to conclude Σ has a model.

The set Γ *does not have a model.*

Proof . We observe that $Th(\mathfrak{N})$, and thus Γ, contains the formula

$$\forall yx[y + x = SSS0 \quad \rightarrow \quad (\; y = 0 \vee y = S0 \vee y = SS0 \vee y = SSS0 \;) \; .$$

Thus, as Γ contains the formula $\exists x[c_i + x = SSS0]$ (for any $i \in \mathbb{Q}$), we have

$$\Gamma \models c_i = 0 \;\vee\; c_i = S0 \;\vee\; c_i = SS0 \;\vee\; c_i = SSS0$$

(for any $i \in \mathbb{Q}$). But we also have $\Gamma \models c_i < c_j$ for any rationals i, j where $i < j$. This makes it easy to see that Γ cannot have a model. (Recall that the set $Th(\mathfrak{N})$ also contains formulas like

$$\forall xyz[x < y \rightarrow \neg x = y] \quad \text{and} \quad \forall xyz[x < y \wedge y < z \rightarrow \neg x < z] \; .$$

If we work a little bit, we can find a Γ-deduction of $\perp$, and then, by Soundness Theorem, we can conclude that Γ does not have a model.)

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Let $\mathfrak{A}$ be an $\mathcal{L}_{NT}$ structure. An infinite sequence $a_0, a_1, a_2, \ldots$ is an $\mathfrak{A}$-*predecessor chain* if $S^{\mathfrak{A}}(a_{i+1}) = a_i$ (for all $i \in \mathbb{N}$). An $\mathfrak{A}$-predecessor chain $a_0, a_1, a_2, \ldots$ *lies below* an $\mathfrak{A}$-predecessor chain $b_0, b_1, b_2, \ldots$ if $a_0 <^{\mathfrak{A}} b_i$ (for all $i \in \mathbb{N}$). If $a_0, a_1, a_2, \ldots$ lies below $b_0, b_1, b_2, \ldots$, then $b_0, b_1, b_2, \ldots$ *lies above* $a_0, a_1, a_2, \ldots$.

Problem 11 *(weight 10 %)*

Prove that $Th(\mathfrak{N})$ has a model $\mathfrak{A}$ such that

- there are infinitely many $\mathfrak{A}$-predecessor chains
- if one $\mathfrak{A}$-predecessor lies below another $\mathfrak{A}$-predecessor chain, then there is an $\mathfrak{A}$-predecessor chain in between, that is, a chain that lies above one of the chains and below the other.

END