UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3600 — Mathematical logic.

Day of examination: Wednesday, December 18, 2019.

Examination hours: 14:30 – 18:30.

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Part I

Let P and Q be unary relation symbols. Let R be a binary relation symbol. Let c be a constant symbol. Let f be a unary function symbol. Furthermore, x and y denote variables.

Problem 1 *(weight 10 %)*

Let $\Sigma = \{\ \neg Qc\,,\ \forall x[Px \to Qx]\ \}$. Give a full Σ-deduction of $\neg\forall x[Px]$. Name all the logical axioms and inference rules involved in the deduction.

———————— Solution:

1.	$\forall x[Px \to Qx]$	Σ
2.	$\forall x[Px \to Qx] \ \to\ [Pc \to Qc]$	(Q1)
3.	$Pc \to Qc$	1, 2, (PC)
4.	$\neg Qc$	Σ
5.	$\neg Pc$	3, 4, (PC)
6.	$\forall x[Px] \ \to\ Pc$	(Q1)
7.	$\neg\forall x[Px]$	5, 6, (PC)

Problem 2 *(weight 10 %)*

Let $\Sigma' = \{\ \neg Qc\,,\ \forall x[Px \to Qx]\,,\ \forall x[Px]\ \}$. Is Σ' consistent? Does Σ' have a model? Give a brief justification of your answers.

———————— Solution:

(Continued on page 2.)

By the previous problem, we have $\Sigma' \vdash \neg\forall x[Px]$. Thus it is easy to see that $\Sigma' \vdash \bot$, and hence Σ' is not consistent. By the Soundness Theorem, Σ' does not have a model.

Problem 3 *(weight 20 %)*

Twenty Questions: Answer each question with a YES or a NO (and nothing else). If you do not answer a question, your answer to that question will be considered as wrong.

1. Does $\forall x[Qx]$ follow tautologically from $\{\ \forall x[Px] \to \forall x[Qx]\,,\ \forall x[Px]\ \}$? **YES**
2. Does $\forall x[Qx]$ follow logically from $\{\ \forall x[Px] \to \forall x[Qx]\,,\ \forall x[Px]\ \}$? **YES**
3. Does Qc follow tautologically from $\{\ \forall x[Px] \to \forall x[Qx]\,,\ \forall x[Px]\ \}$? **NO**
4. Does Qc follow logically from $\{\ \forall x[Px] \to \forall x[Qx]\,,\ \forall x[Px]\ \}$? **YES**
5. Does $\forall x[Px \to Qx]$ follow logically from $\{\ \forall x[Px] \to \forall x[Qx]\,,\ \forall x[Px]\ \}$? **YES**
6. Does $\forall x[Px] \to \forall x[Qx]$ follow logically from $\{\ \forall x[Px \to Qx]\,,\ \forall x[Px]\ \}$? **YES**
7. Does $\forall x[Px \to Qx]$ follow logically from $\{\ \forall x[Px] \to \forall x[Qx]\ \}$? **NO**
8. Does $\forall x[Px] \to \forall x[Qx]$ follow logically from $\{\ \forall x[Px \to Qx]\ \}$? **YES**
9. Does $\exists y \forall x[Rxy]$ follow logically from $\{\ \forall x[Rxfx]\ \}$? **NO**
10. Does $\forall x \exists y[Rxy]$ follow logically from $\{\ \forall x[Rxfx]\ \}$? **YES**
11. Does $\exists y \forall x[Rxy]$ follow logically from $\{\ \forall x[Rxc]\ \}$? **YES**
12. Does $\forall x \exists y[Rxy]$ follow logically from $\{\ \forall x[Rxc]\ \}$? **YES**
13. Does $Qf(c)$ follow tautologically from $\{\ \forall x[Px \to Qx]\,,\ \forall x[Px] \to \forall x[Qx]\ \}$? **NO**
14. Does $Qf(c)$ follow logically from $\{\ \forall x[Px \to Qx]\,,\ \forall x[Px] \to \forall x[Qx]\ \}$? **NO**
15. Does $Pc \to \forall x[Qx]$ follow logically from $\{\ Pc \to Qx\ \}$? **YES**
16. Does $Px \to \forall x[Qx]$ follow logically from $\{\ Px \to Qx\ \}$? **NO**
17. Does $\exists x[Px] \to \forall x[Qx]$ follow logically from $\{\ Px \to \forall x[Qx]\ \}$? **YES**
18. Does $x = x$ follow logically from $\emptyset$? **YES**
19. Does $x = y$ follow logically from $\emptyset$? **NO**
20. Does $\neg x = y$ follow logically from $\emptyset$? **NO**

(Continued on page 3.)

Part II

Let $\mathcal{L}$ be the first-order language $\{\preceq, f, c\}$ where $\preceq$ is a binary relation symbol, f is a binary function symbol and c is a constant symbol. Let T be the $\mathcal{L}$-theory consisting of the non-logical axioms

(T_1) $\forall xy[\ \neg\, c = f(x,y)\ ]$

(T_2) $\forall x_1x_2y_1y_2[\ f(x_1,x_2) = f(y_1,y_2)\ \rightarrow\ (x_1 = y_1 \wedge x_2 = y_2)\ ]$

(T_3) $\forall x[\ x \preceq c\ \leftrightarrow\ x = c\ ]$

(T_4) $\forall xy_1y_2[\ x \preceq f(y_1,y_2)\ \leftrightarrow\ (\ x = f(y_1,y_2)\ \vee\ x \preceq y_1\ \vee\ x \preceq y_2\ )\ ]$.

Problem 4 *(weight 10 %)*

Show that

$$T \vdash \neg f(c,c) = f(f(c,c),c)\ .$$

Sketch a formal deduction.

——————————— Solution:

1.	$\forall xy[\ \neg\, c = f(x,y)\ ]\ \rightarrow\ \forall y[\ \neg\, c = f(c,y)\ ]$	(Q1)
2.	$\forall y[\ \neg\, c = f(c,y)\ ]\ \rightarrow\ \neg\, c = f(c,c)$	(Q1)
3.	$\forall xy[\ \neg\, c = f(x,y)\ ]$	(T_1)
4.	$\neg\, c = f(c,c)$	1, 2, 3 and (PC)

This shows that

$$T \vdash \neg c = f(c,c) \qquad (*)$$

In a similar way, by using (T_2), (Q1) and (PC), we can show that

$$T \vdash f(c,c) = f(f(c,c),c)\ \rightarrow\ (c = f(c,c)\ \wedge\ c = c)\ . \qquad (**)$$

By (*), (**) and (PC), we have

$$T \vdash \neg f(c,c) = f(f(c,c),c)\ .$$

Problem 5 *(weight 10 %)*

Show that

$$T \vdash \neg s = t\ .$$

for any variable-free $\mathcal{L}$-terms s, t where $s \neq t$ (so s and t are not syntactically equal). Use induction on the structure of s.

——————————— Solution:

The proof splits into the cases: $s :\equiv c$ and $s :\equiv f(s_1, s_2)$.

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Case $s :\equiv c$. Assume $s \not\equiv t$. Then $t :\equiv f(t_1, t_2)$. By (T_1), we have $T \vdash \neg s = t$.

Case $s :\equiv f(s_1, s_2)$. Assume $s \not\equiv t$. The proof splits into the subcases $t :\equiv c$ and $t :\equiv f(t_1, t_2)$.

If $t :=\equiv c$, we have $T \vdash \neg s = t$ by (T_1).

We turn to the case $t :\equiv f(t_1, t_2)$. As s and t are different terms, we can conclude that s_1 is different from t_1 or s_2 is different from t_2. We can without loss of generality assume that s_1 is differ net from t_1 (so the case when s_2 is different from t_2 is similar). By our induction hypothesis we have

$$T \vdash \neg s_1 = t_1 \tag{i}$$

By (T_2), we have

$$T \vdash f(s_1, s_2) = f(t_1, t_2) \ \rightarrow \ (s_1 = t_1 \land s_2 = t_2) \tag{ii}$$

By (i), (ii) and (PC), we have $T \vdash \neg f(s_1, s_2) = f(t_1, t_2)$.

> **Lemma 1.** For any variable-free $\mathcal{L}$-terms s and t, we have $T \vdash s \preceq t$ or $T \vdash \neg s \preceq t$.

Problem 6 *(weight 10 %)*

Prove Lemma 1. Use induction on the structure of t.

——————————— Solution:

The proof splits into the cases $t :\equiv c$ and $t :\equiv f(t_1, t_2)$.

Case $t :\equiv c$. If $s :\equiv c$, then we have $T \vdash s \preceq t$ by (T_3). If $s :\not\equiv c$, then we have $T \vdash \neg s \preceq t$ by (T_3) and Problem 5.

Case $t :\equiv f(t_1, t_2)$. First we observe that we have

$$T \vdash s \preceq f(t_1, t_2) \ \leftrightarrow \ (\ s = f(t_1, t_2) \ \lor \ s \preceq t_1 \ \lor \ s \preceq t_2\ ) \tag{iii}$$

by (T_4). Next we observe that if s is the same term as $f(t_1, t_2)$, then we have $T \vdash s \preceq f(t_1, t_2)$ by (iii), (E1) and (PC). A short explanation: we have $\vdash t = t$, and thus also $T \vdash t = t$, for any term t.

Thus, we conclude that the theorem holds when s and $f(t_1, t_2)$ are the same term. We are left to prove that the theorem holds when s and $f(t_1, t_2)$ are different terms. So we assume that s and $f(t_1, t_2)$ are different terms. By Problem 5 we have

$$T \vdash \neg s = f(t_1, t_2) \tag{iv}$$

The induction hypothesis applied to t_1 yields

$$T \vdash s \preceq t_1 \quad \text{or} \quad T \vdash \neg s \preceq t_1 \ .$$

(Continued on page 5.)

The induction hypothesis applied to t_2 yields

$$T \vdash s \preceq t_2 \quad \text{or} \quad T \vdash \neg s \preceq t_2 \,.$$

The proof splits into the two cases

$$\text{at least one of } T \vdash s \preceq t_1 \text{ and } T \vdash s \preceq t_2 \text{ holds} \tag{A}$$

and

$$\text{neither } T \vdash s \preceq t_1 \text{ nor } T \vdash s \preceq t_2 \text{ holds.} \tag{B}$$

In case (A), we have $T \vdash s \preceq f(t_1, t_2)$ by (iii) and (PC).

We turn to case (B). Since we neither $T \vdash s \preceq t_1$ nor $T \vdash s \preceq t_2$, it must be the case that both $T \vdash \neg s \preceq t_1$ and $T \vdash \neg s \preceq t_2$ holds. By (iii), (iv) and (PC), we have $T \vdash \neg s \preceq f(t_1, t_2)$.

Problem 7 *(weight 10 %)*

Let ϕ be a quantifier-free and variable-free $\mathcal{L}$-formula. Prove that we have $T \vdash \phi$ or $T \vdash \neg\phi$. Use Lemma 1.

——————————— Solution:

Assume ϕ is an atomic formula, that is, ϕ is of the form $s = t$ or of the form $s \preceq t$. Then we have $T \vdash \phi$ or $T \vdash \neg\phi$ by Problem 5 and Problem 6. (If s and t are the same term, then we have $T \vdash s = t$ by (E1) and other logical axioms.)

Assume $\phi :\equiv \alpha \vee \beta$. By our induction hypothesis, we have

$$T \vdash \alpha \quad \text{or} \quad T \vdash \neg\alpha$$

and

$$T \vdash \beta \quad \text{or} \quad T \vdash \neg\beta \,.$$

If $T \vdash \alpha$, we have $T \vdash \alpha \vee \beta$ by (PC). If $T \vdash \beta$, we have $T \vdash \alpha \vee \beta$ by (PC). Otherwise, that is, if we neither have $T \vdash \alpha$ nor $T \vdash \beta$, then we have both $T \vdash \neg\alpha$ and $T \vdash \neg\beta$, and thus, by (PC), we have $\neg(\alpha \vee \beta)$.

Assume $\phi :\equiv \neg\alpha$. By our induction hypothesis, we have

$$T \vdash \alpha \quad \text{or} \quad T \vdash \neg\alpha$$

and thus, by (PC), we have

$$T \vdash \neg\neg\alpha \quad \text{or} \quad T \vdash \neg\alpha \,.$$

Problem 8 *(weight 10 %)*

Do we have $T \vdash \forall x[\neg x = f(x, x)]$? Justify your answer.

We say that an $\mathcal{L}$-structure $\mathfrak{A}$ is *ill-founded* if its universe contains elements $a_0, a_1, a_2, \ldots$ such that $a_{i+1} \neq a_i$ and $a_{i+1} \preceq^{\mathfrak{A}} a_i$ (for all $i \in \mathbb{N}$).

(Continued on page 6.)

Problem 9 *(weight 10 %)*

Explain why any consistent extension of T has an ill-founded model.

——————————— Solution:

Let T' be a consistent extension of T. By the Completeness Theorem, T' has a model $\mathfrak{B}$. Let $t_0 :\equiv c$ and $t_{n+1} :\equiv f(t_n, c)$. By the problems above, we have $T \vdash t_i \preceq t_{i+1}$ and $T \vdash \neg t_i = t_{i+1}$ (for all i), and thus, by the Soundness Theorem, we have $\mathfrak{B} \models t_i \preceq t_{i+1}$ and $\mathfrak{B} \models \neg t_i = t_{i+1}$.

Let $\mathcal{L}_*$ be $\mathcal{L}$ extended by infinitely many fresh constant symbols $c_0, c_1, c_2, \ldots$. Let $\Gamma_0 = Th(\mathfrak{B})$ and $\Gamma_{n+1} = \Gamma_n \cup \{c_{n+1} \preceq c_n\}$. Furthermore, let $\Gamma = \bigcup_i \Gamma_i$.

First we argue that Γ_n has a model $\mathfrak{A}_n$.

Let $\mathfrak{A}_n$ be $\mathfrak{B}$ extended by $c_i^{\mathfrak{A}_n} = t_{n-i}^{\mathfrak{B}}$, for $i = 0, \ldots, n$, and $c_j^{\mathfrak{A}_n} = t_0^{\mathfrak{B}}$, for $j > n$. Then $\mathfrak{A}_n \models \Gamma_n$.

Next we argue that every finite subset of Γ has a model: Let Ω be a finite subset of Γ. Then we have $\Omega \subseteq \Gamma_n$ for some sufficiently large n, and hence, $\mathfrak{A}_n \models \Omega$.

This proves that every finite subset of Γ has a model. By the Compactness Theorem, Γ has a model $\mathfrak{A}$. The reduct of $\mathfrak{A}$ to the language $\mathcal{L}$ is an ill-founded model for T'. The model is ill-founded as we have $c_{i+1}^{\mathfrak{A}} \preceq^{\mathfrak{A}} c_i^{\mathfrak{A}}$ and $c_{i+1}^{\mathfrak{A}} \neq c_i^{\mathfrak{A}}$ (for all $i \in \mathbb{N}$).

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