UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT-INF3600 — Mathematical logic.
Day of examination:	Monday, December 19, 2022.
Examination hours:	15:00 – 19:00.

This problem set consists of 7 pages.

Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is
complete before you attempt to answer anything.

The weights might be adjusted.

PART I

Let Q be a unary relation symbol, let S be a unary function symbol, let 0 be a constant symbol and let $\mathcal{L}$ be the language $\{0, S, Q\}$. Furthermore, let

$$\Sigma_1 = \{ \forall x[Q(Sx) \to Q(x)] , Q(S0) \} .$$

Problem 1 *(weight 10 %)*

Give a full Σ_1-deduction of $Q(0)$. Name all the logical axioms and inference rules involved in the deduction.

——— Solution:

1. $\forall x[Q(Sx) \to Q(x)] \to (Q(S0) \to Q(0))$ (Q1)
2. $\forall x[Q(Sx) \to Q(x)]$ Σ_1
3. $Q(S0) \to Q(0)$ 1, 2, (PC)
4. $Q(S0)$ Σ_1
5. $Q(0)$ 3, 4, (PC)

Problem 2 *(weight 10 %)*

Give a full Σ_1-deduction of $0 = x \to Q(x)$. Name all the logical axioms and inference rules involved in the deduction.

——— Solution:

(Continued on page 2.)

$\vdots$

as above

$\vdots$

5.	$Q(0)$	3, 4, (PC)
6.	$y = x \to (Q(y) \to Q(x))$	(E3)
7.	$\top \to [y = x \to (Q(y) \to Q(x))]$	6, (PC)
8.	$\top \to \forall y[y = x \to (Q(y) \to Q(x))]$	7, (QR)
9.	$\forall y[y = x \to (Q(y) \to Q(x))]$	8, (PC)
10.	$\forall y[y = x \to (Q(y) \to Q(x))] \to [0 = x \to (Q(0) \to Q(x))]$	(Q1)
11.	$0 = x \to (Q(0) \to Q(x))$	9, 10, (PC)
12.	$0 = x \to Q(x)$	5, 11, (PC)

For any natural number n, we define the numeral $\overline{n}$ by $\overline{0} = 0$ and $\overline{n+1} = S\overline{n}$. Let

$$\Sigma_n = \{ \forall x[Q(Sx) \to Q(x)], Q(\overline{n}) \} .$$

Problem 3 *(weight 10 %)*

Do we have $\Sigma_{17} \models \Sigma_{16}$? Do we have $\Sigma_{16} \models \Sigma_{17}$? Justify your answers.

———————————— Solution:

Do we have $\Sigma_{17} \models \Sigma_{16}$? YES. In order to justify our answer we have to argue that any model for Σ_{17} will also be a model for Σ_{16}. This will be true since $\Sigma_{17} \models Q(\overline{16})$, that is, $Q(\overline{16})$ follows (logically) from $\forall x[Q(Sx) \to Q(x)]$ and $Q(\overline{17})$.

Do we have $\Sigma_{16} \models \Sigma_{17}$? NO. In order to justify our answer we have to argue that it is not true that any model for Σ_{16} also is a model for Σ_{17}. We give a structure that is a model for Σ_{16}, but not for Σ_{17}: Let $\mathfrak{A}$ be the structure where the universe is the set of natural numbers $\{0, 1, 2, \ldots\}$. Let $0^{\mathfrak{A}} = 0$, let $S^{\mathfrak{A}}(x) = x + 1$ and let $Q^{\mathfrak{A}} = \{ 0, 1, 2, 3, \ldots, 16\}$. Then we have $\mathfrak{A} \models \Sigma_{16}$ and $\mathfrak{A} \not\models \Sigma_{17}$.

Let $\Sigma = \bigcup_{i \in \mathbb{N}} \Sigma_i$.

Problem 4 *(weight 10 %)*

Give an $\mathcal{L}$-sentence ϕ sucht that $\Sigma \nvdash \phi$ and $\Sigma \nvdash \neg\phi$. Prove that we indeed have $\Sigma \nvdash \phi$ and $\Sigma \nvdash \neg\phi$.

———————————— Solution:

(Continued on page 3.)

Let $\phi :\equiv \forall x[Q(x)]$.

Let $\mathfrak{A}$ be the $\mathcal{L}$-structure where the universe is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $0^{\mathfrak{A}} = 0$, let $S^{\mathfrak{A}}(x) = x + 1$ and let $Q^{\mathfrak{A}} = \mathbb{N}$. Then we have $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \models \phi$. By the Soundness Theorem, we have $\Sigma \not\vdash \neg\phi$.

Let $\mathfrak{B}$ be the $\mathcal{L}$-structure where the universe $\mathbb{N} \cup \{\omega\}$ (the natural numbers extended by an element ω). Let $0^{\mathfrak{A}} = 0$; let $S^{\mathfrak{A}}(\omega) = \omega$ and let $S^{\mathfrak{A}}(x) = x + 1$ for any $x \in \mathbb{N}$; furthermore, let $Q^{\mathfrak{A}} = \mathbb{N}$. Then we have $\mathfrak{A} \models \Sigma$ and $\mathfrak{A} \models \neg\phi$. By the Soundness Theorem, we have $\Sigma \not\vdash \phi$.

PART II

Let f be a binary function symbol, let c be a constant symbol and let $\mathcal{L}$ be the language $\{c, f\}$. Let

- $\Gamma_1 = \{ \forall x[\ x = c\ ],\ f(c,c) = c \}$
- $\Gamma_2 = \{ \forall x[\ x = c\ ],\ \neg(f(c,c) = c) \}$
- $\Gamma_3 = \{ \forall x[\ \neg(x = c)\ ],\ \neg(f(c,c) = c) \}$
- $\Gamma_4 = \{ \neg\forall x[\ x = c\ ],\ f(c,c) = c \}$.

Problem 5 *(weight 10 %)*

What does it mean that a set of first-order formulas is consistent? (Give the definition.) Is Γ_1 consistent? Is Γ_2 consistent? Is Γ_3 consistent? Is Γ_4 consistent? Justify your answers.

——————————— Solution:

A set Σ of first-order formulas is consistent iff $\Sigma \not\vdash \bot$. By the Soundness and Completeness theorem, Σ has a modell iff $\Sigma \not\vdash \bot$.

- $\Gamma_1 = \{\forall x[\ x = c\ ],\ f(c,c) = c\}$ is consistent. It is easy to see that the set has a model, in fact the set has one model up to isomorphism, the universe of that model contains exactly one element.
- $\Gamma_2 = \{\forall x[\ x = c\ ], \neg(f(c,c) = c)\}$ is not consistent. It is easy to see that $\Gamma_2 \vdash \exists x[\neg x = c]$ (use the axiom (Q2), and then, by logical axioms, $\Gamma_2 \vdash \neg\forall x[\ x = c\ ]$. Thus, $\Gamma_2 \vdash \bot$ (since we also have $\Gamma_2 \vdash \forall x[\ x = c\ ]$).
- $\Gamma_3 = \{\forall x[\ \neg(x = c)\ ],\ \neg(f(c,c) = c)\}$ is not consistent. The set is not consistent since the sentence $\forall x[\ \neg(x = c)\ ]$ does not have model (the sentence is false in any structure as some element of the universe has to be the interpretation of the constant symbol c).
- $\Gamma_4 = \{\neg\forall x[\ x = c\ ],\ f(c,c) = c\}$ is consistent. It is easy to see that Γ_4 has a model.

For any $\mathcal{L}$-term t, let $(t)^x_y$ denote the term t where every occurrence of the variable x is replaced by the variable y. E.g., $(ffv_1cfv_2v_1)^{v_1}_{v_3}$ denotes the term $ffv_3cfv_2v_3$ and $(fcc)^{v_3}_{v_7}$ denotes the term fcc.

(Continued on page 4.)

Theorem 1. For any variables x, y and any $\mathcal{L}$-term t, we have

$$\vdash x = y \ \rightarrow \ t = (t)^x_y \ .$$

Problem 6 *(weight 10 %)*

Prove Theorem 1. Use induction on the structure of the term t.

——————————— Solution:

Case t is a variable. The case splits into two subcases: (i) t is the variable x and (ii) t is a variable different from x. First we deal with case (i). We have $t :\equiv x$ and $(t)^x_y :\equiv y$. Thus, we need to prove that

$$\vdash x = y \rightarrow x = y \ . \tag{1}$$

The propositional form of the formula $\vdash x = y \rightarrow x = y$ is $A \rightarrow A$. Thus, (1) holds since $A \rightarrow A$ is a tautology. Next we deal with case (ii). We have $t :\equiv z$ and $(t)^x_y :\equiv z$. Thus, we need to prove that

$$\vdash x = y \rightarrow z = z \ . \tag{2}$$

(Note that z is different from x, but z might be y. Our proof works no matter if z is y or not.) Observe that $z = z$ is an instance of (E1), and then one application of (PC) yields $x = y \rightarrow z = z$. This proves that (2) holds.

Case $t :\equiv c$. We need to prove that

$$\vdash x = y \rightarrow c = c \ . \tag{3}$$

This case is simmilar to the case when t is a variable different from x. By using (E1) (and other logical axioms), we can deduce $x = y \rightarrow c = c$.

Case $t :\equiv ft_1t_2$. We need to prove

$$\vdash x = y \ \rightarrow \ ft_1t_2 = (ft_1t_2)^x_y \ . \tag{4}$$

The induction hypothesis yields

$$\vdash x = y \rightarrow t_1 = (t_1)^x_y \quad \text{and} \quad \vdash x = y \rightarrow t_2 = (t_2)^x_y \ . \tag{5}$$

By (E2) and other logical axioms, we have

$$\vdash t_1 = (t_1)^x_y \ \wedge \ t_2 = (t_2)^x_y \ \rightarrow \ ft_1t_2 = f(t_1)^x_y(t_2)^x_y \tag{6}$$

By (5), (6) and (PC), we have

$$\vdash x = y \ \rightarrow \ ft_1t_2 = f(t_1)^x_y(t_2)^x_y$$

and thus (4) holds since $(ft_1t_2)^x_y :\equiv f(t_1)^x_y(t_2)^x_y$.

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(Continued on page 5.)

Problem 7 *(weight 10 %)*

Do we have $\vdash t = (t)^x_y \rightarrow x = y$ (for any variables x, y and any $\mathcal{L}$-term t)? Do we have $\vdash \neg(t = (t)^x_y \rightarrow x = y)$ (for any variables x, y and any $\mathcal{L}$-term t)? Justify your answers.

——————————— Solution:

Do we have $\vdash t = (t)^x_y \rightarrow x = y$ (for any variables x, y and any $\mathcal{L}$-term t)? NO. By the Soundness Theorem, we have

$$\vdash t = (t)^x_y \rightarrow x = y \quad \Rightarrow \quad \models t = (t)^x_y \rightarrow x = y \ . \tag{7}$$

Let $\mathfrak{A}$ a be an $\mathcal{L}$-structure with at least two elements in the universe. Let $t :\equiv c$ and let s be an assignment such that $s(x) \neq s(y)$. We have $\mathfrak{A} \not\models c = c \rightarrow x = y\,[s]$. Thus, we have $\not\models c = c \rightarrow x = y$. By (7), we have $\nvdash c = c \rightarrow x = y$.

Do we have $\vdash \neg(t = (t)^x_y \rightarrow x = y)$ (for any variables x, y and any $\mathcal{L}$-term t)? NO. Let $\mathfrak{A}$ a be any $\mathcal{L}$-structure. Let $t :\equiv c$ and let s be an assignment such that $s(x) = s(y)$. We have $\mathfrak{A} \not\models \neg(c = c \rightarrow x = y)\,[s]$. Thus, we have $\not\models c = c \rightarrow x = y$. By (7), we have $\nvdash \neg(c = c \rightarrow x = y)$.

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PART III

Recall the language $\mathcal{L}_{NT}$, that is, the language $\{0, S, +, \cdot, E, <\}$, and its standard structure $\mathfrak{N}$. Some $\mathcal{L}_{NT}$-formulas contain bounded quantifiers, and some $\mathcal{L}_{NT}$-formulas are Δ-formulas.

Problem 8 *(weight 5 %)*

What is a bounded quantifier? What is a Δ-formula?

Let $\phi(x) :\equiv (\exists y)[y + y = x]$, and let $\psi(x) :\equiv \neg(\exists y)[y + y = x]$.

Problem 9 *(weight 5 %)*

Give a Δ-formula $\phi_0(x)$ such that $\mathfrak{N} \models \phi_0(\overline{a})$ if and only if $\mathfrak{N} \models \phi(\overline{a})$ (for any natural number a). Give a Δ-formula $\psi_0(x)$ such that $\mathfrak{N} \models \psi_0(\overline{a})$ if and only if $\mathfrak{N} \models \psi(\overline{a})$ (for any natural number a).

——————————— Solution:

$$\phi_0(x) :\equiv \ (\exists y < Sx)[y + y = x] \quad \text{and} \quad \psi_0(x) :\equiv \ (\forall y < Sx)[\neg(y + y = x)] \ .$$

———————————

Let p_i denote the i'th prime, that is, $p_1 = 2$ and $p_2 = 3$ and so on. We encode a nonempty finite sequence of natural numbers $a_1, a_2, \ldots, a_k$ as the single natural number $\langle a_1, a_2, \ldots, a_k \rangle$ where

$$\langle a_1, a_2, \ldots, a_k \rangle = p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot \ldots \cdot p_k^{a_k+1} \ .$$

(Continued on page 6.)

The Δ-formula $\mathit{IthElement}(x_1, x_2, x_3)$ is known from Leary & Kristiansen's textbook. We have $\mathfrak{N} \models \mathit{IthElement}(\overline{b}, \overline{i}, \overline{a})$ if and only if there exists a sequence of natural numbers $a_1, a_2, \ldots, a_k$ such that $a = \langle a_1, a_2, \ldots, a_k \rangle$ and $a_i = b$.

We define the sequence $F_0, F_1, F_2, \ldots$ of Fibonacci numbers by $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

Problem 10 *(weight 10 %)*

Give an $\mathcal{L}_{NT}$-formula $\theta(x_1, x_2)$ such that $\mathfrak{N} \models \theta(\overline{i}, \overline{m})$ if and only if $F_i = m$. Use the formula $\mathit{IthElement}(x_1, x_2, x_3)$ to construct $\theta(x_1, x_2)$.

———————————— Solution:

We write *IthE* in place of *IthElement* due to the with of an A4 paper (which is 11.7 inches).

$$
\begin{aligned}
\theta(x_0, x_1) :\equiv (\exists w)\big[\ & \mathit{IthE}(\overline{1}, \overline{1}, w) \ \land\ \mathit{IthE}(\overline{1}, \overline{2}, w) \\
& \land\ (\forall i < x_0)(\forall y)(\forall y')(\forall y'')\big[\ \big(\ \mathit{IthE}(y, i, w) \ \land\ \mathit{IthE}(y', i + \overline{1}, w) \ \land \\
& \qquad \mathit{IthE}(y'', i + \overline{2}, w)\ \big) \ \rightarrow\ y'' = y + y'\ \big] \\
& \qquad\qquad \land\ \mathit{IthE}(x_1, x_0 + \overline{1}, w)\ \big]
\end{aligned}
$$

Explanation : The formula states that there exists a number w that encodes the sequence

$$\langle F_0, F_1, F_2, \ldots, F_{x_0}, F_{x_0+1} \rangle$$

and that x_1 is the second but last element in the sequence encoded by w.

Recall the first-order theory N from Leary & Kristiansen's book (N is given by 11 nonlogical $\mathcal{L}_{NT}$-axioms).

Problem 11 *(weight 10 %)*

Give an $\mathcal{L}_{NT}$-formula $\eta(x_1, x_2)$ such that

(1) $N \vdash \eta(\overline{i}, \overline{m})$ if $F_i = m$

(2) $N \vdash \neg\eta(\overline{i}, \overline{m})$ if $F_i \neq m$

and explain why (1) and (2) hold.

———————————— Solution:

Let $\eta(x_1, x_2)$ be of the form

$$
\begin{aligned}
\eta(x_0, x_1) :\equiv (\exists w < t(x_0)\big[\ & \mathit{IthE}(\overline{1}, \overline{1}, w) \ \land\ \mathit{IthE}(\overline{1}, \overline{2}, w) \\
& \land\ (\forall i < x_0)(\forall y < w)(\forall y' < w)(\forall y'' < w)\big[\ \big(\ \mathit{IthE}(y, i, w) \ \land\ \mathit{IthE}(y', i + \overline{1}, w) \ \land \\
& \qquad \mathit{IthE}(y'', i + \overline{2}, w)\ \big) \ \rightarrow\ y'' = y + y'\ \big] \\
& \qquad\qquad \land\ \mathit{IthE}(x_1, x_0 + \overline{1}, w)\ \big]
\end{aligned}
$$

(Continued on page 7.)

where the bound $t(x_0)$ is an $\mathcal{L}_{NT}$-term with only x_0 free. If the bound $t(x_0)$ have the property

$$\langle F_0, F_1, F_2, \ldots, F_{x_0}, F_{x_0+1}\rangle \;<\; t^{\mathfrak{N}}(x_0) \tag{8}$$

then we have $\mathfrak{N} \models \theta(\overline{a}, \overline{b})$ iff $\mathfrak{N} \models \eta(\overline{a}, \overline{b})$ (for any natural numers a and b). Now, we have $p_i \leq 2^i$ (for any $i \geq 1$, this is a very tight bound on the ith prime, in a classical texbook on number theory the bound will be given by $p_i < 2^{2^i}$). Given such a bound in the ith prime, it is not hard to prove that (8) holds if $t(x_0)$ is something like

$$t(x_0) :\equiv \overline{2}E(\overline{2}E(\overline{2}(\overline{2}E(\overline{2}E(\overline{2}E(\overline{2}E(\overline{2}E(\overline{2}E(\overline{2}Ex_0)))))))))$$

and which can be written up in less than 11.7 inches (the with of an A4 paper).

We observe that $\eta(x_0, x_1)$ is a Δ-formula, and thus (by results proved in Leary & Kristiansen), we have

$$\mathfrak{N} \models \eta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad N \vdash \eta(\overline{i}, \overline{m})$$

and

$$\mathfrak{N} \models \neg\eta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad N \vdash \neg\eta(\overline{i}, \overline{m})\;.$$

This shows that clause (1) of problem 11 holds as we have

$$F_i = M \quad\Leftrightarrow\quad \mathfrak{N} \models \theta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad \mathfrak{N} \models \eta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad N \vdash \eta(\overline{i}, \overline{m})\;.$$

Furthermore, that clause (2) of problem 11 holds since

$$F_i \neq M \quad\Leftrightarrow\quad \mathfrak{N} \not\models \theta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad \mathfrak{N} \not\models \eta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad \mathfrak{N} \models \neg\eta(\overline{i}, \overline{m}) \quad\Leftrightarrow\quad N \vdash \neg\eta(\overline{i}, \overline{m})\;.$$

END