Triangular Bézier surfaces

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We study Bézier surfaces defined over triangular domains.

1 Polynomial spaces

We have seen how spaces of bivariate polynomials can be constructed as tensor-products of the univariate ones. An alternative choice of polynomial space is, for each $n \geq 0$, the space of polynomials of the form

$$p(x, y) = \sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j,$$

where we understand that $i \geq 0$ and $j \geq 0$ in the summation. As in the univariate case, we denote this space by π_n. Such a polynomial has degree $\leq n$, its degree (sometimes called the total degree) being the largest value of $i + j$ over all non-zero $a_{i,j}$ in the summation. The monomials $x^i y^j$ in the sum are linearly independent and therefore form a basis of π_n. Since the number of such polynomials is

$$(n + 1) + n + \cdots + 1 = \binom{n+2}{2},$$

this is also the dimension of π_n. For example, the monomial basis of π_2 is

$$\{1, x, y, x^2, xy, y^2\},$$

which has six elements.

If we use such polynomials for modelling surfaces it turns out that there is again an alternative basis that makes the modelling easier. This is the basis of bivariate Bernstein polynomials

$$B^n_{i,j}(x,y) = \frac{n!}{i!j!(n-i-j)!} x^i y^j (1-x-y)^{n-i-j}, \qquad 0 \le i+j \le n,$$

To show that these polynomials are linearly independent, suppose that

$$\sum_{0 \le i+j \le n} c_{i,j} x^i y^j (1-x-y)^{n-i-j} = 0 \qquad \text{for } x,y > 0 \text{ and } x+y < 1.$$

Then, setting $u = x/(1-x-y)$ and $v = y/(1-x-y)$ it follows that

$$\sum_{0 \le i+j \le n} c_{i,j} u^i v^j = 0 \qquad \text{for } u,v > 0,$$

which implies that the coefficients $c_{i,j}$ are all zero. We usually only consider points (x,y) in the triangular domain

$$D := \{(x,y) \in \mathbb{R}^2 : x,y \ge 0, x+y \le 1\}.$$

The polynomials $B^n_{i,j}$ are non-negative in D and positive in the interior of D. By the binomial theorem they also sum to one,

$$\sum_{0 \le i+j \le n} B^n_{i,j}(x,y) = \big(x + y + (1-x-y)\big)^n = 1.$$

The form of the Bernstein polynomial $B^n_{i,j}$ clearly suggests another way of viewing it, as a function of the three variables x, y, and $1-x-y$, rather than just x and y. If we make the definition

$$B^n_{i,j,k}(u,v,w) = \frac{n!}{i!j!k!} u^i v^j w^k, \qquad i+j+k = n, \tag{1}$$

then we see that

$$B^n_{i,j}(x,y) = B^n_{i,j,n-i-j}(x,y,1-x-y).$$

For example, with $n = 3$ there are 10 polynomials,

$$\begin{array}{ccccccc}
 & & & B^3_{003} & & & \\
 & & B^3_{102} & & B^3_{012} & & \\
 & B^3_{201} & & B^3_{111} & & B^3_{021} & \\
B^3_{300} & & B^3_{210} & & B^3_{120} & & B^3_{030}
\end{array}$$

given by the formulas

$$\begin{array}{ccccccc} & & & w^3 & & & \\ & & 3uw^2 & & 3vw^2 & & \\ & 3u^2w & & 6uvw & & 3v^2w & \\ u^3 & & 3u^2v & & 3uv^2 & & v^3 \end{array}$$

We can also simplify notation in (1) by defining $\mathbf{i} = (i, j, k)$ and $\mathbf{u} = (u, v, w)$, so that we can write

$$B^n_{i,j,k}(u, v, w) = B^n_{\mathbf{i}}(\mathbf{u}).$$

2 Triangular Bézier surfaces

We define a triangular Bézier surface in $\mathbb{R}^d$ as a parametric polynomial surface whose parameter domain is some triangle $T \subset \mathbb{R}^2$. Suppose this triangle T has vertices $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$. Then, for a given point $\mathbf{t} \in T$, we let

$$\mathbf{p}(\mathbf{t}) = \sum_{|\mathbf{i}|=n} \mathbf{c}_{\mathbf{i}} B^n_{\mathbf{i}}(\mathbf{u}), \qquad \mathbf{c}_{\mathbf{i}} \in \mathbb{R}^d, \tag{2}$$

where $\mathbf{i} = (i, j, k)$, $|\mathbf{i}| = i + j + k$, $\mathbf{u} = (u, v, w)$, and the values $u, v, w \in \mathbb{R}$ are the *barycentric coordinates* of the point $\mathbf{t}$ with respect to the triangle T, i.e., the three values such that

$$u + v + w = 1, \tag{3}$$

$$u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3 = \mathbf{t}. \tag{4}$$

The points $\mathbf{c}_{\mathbf{i}}$ are the *control points* of $\mathbf{p}$, which, together with all line segments that connect neighbouring points, form the *control net* of $\mathbf{p}$. Figure 1 shows a quadratic surface, where $n = 2$, with its control net.

How do we find u, v, w? If $\mathbf{t} = (s, t)$ and $\mathbf{a}_k = (a_k, b_k)$, we can express (3–4) in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ s \\ t \end{bmatrix},$$

and Cramer's rule gives the solution

$$u = D_1/D, \qquad v = D_2/D, \qquad w = D_3/D,$$

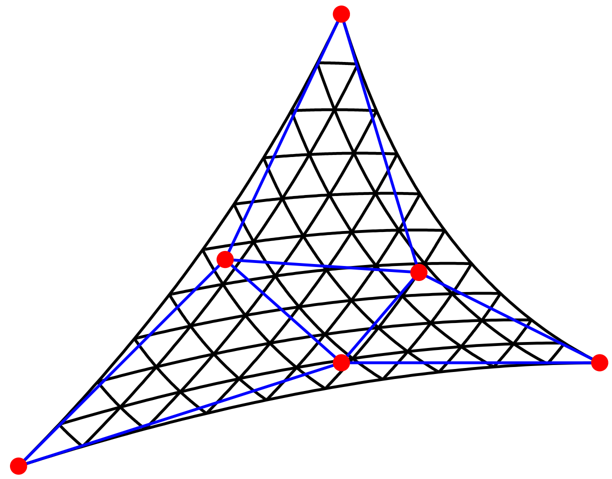

Figure 1: A quadratic Bézier surface

where

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ s & a_2 & a_3 \\ t & b_2 & b_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & s & a_3 \\ b_1 & t & b_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & s \\ b_1 & b_2 & t \end{vmatrix},$$

and

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Triangular Bézier surfaces have some properties analogous to tensor-product ones. For example, on each of the three sides of the triangle T, the surface $\mathbf{p}$ is a Bézier curve whose control polygon is the corresponding boundary polygon of the control net of $\mathbf{p}$. For example, if the point $\mathbf{t}$ is a point on the edge $[\mathbf{a}_1, \mathbf{a}_2]$ then its barycentric coordinates have the property that $u + v = 1$ and $w = 0$, and so

$$\mathbf{p}(\mathbf{t}) = \mathbf{p}(u\mathbf{a}_1 + v\mathbf{a}_2) = \sum_{i+j=n} \mathbf{c}_{i,j,0} \frac{n!}{i!j!} u^i v^j.$$

At the corners of the triangle T, the surface equals one of the corner control points. For example if $\mathbf{t} = \mathbf{a}_1$ then $u = 1$ and $v = w = 0$ and so

$$\mathbf{p}(\mathbf{t}) = \mathbf{p}(\mathbf{a}_1) = \mathbf{c}_{n,0,0}.$$

Like tensor-product Bézier surfaces, triangular Bézier surfaces are affinely invariant and have the convex hull and bounding box properties.

3 The de Casteljau algorithm

If we replace (u, v, w) in (1) by (u_1, u_2, u_3) and we define

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1),$$

it is easy to write down a recursion formula for the $B^n_{\mathbf{i}}$, analogous to the univariate case,

$$B^n_{\mathbf{i}}(\mathbf{u}) = \sum_{k=1}^{3} u_k B^{n-1}_{\mathbf{i}-\mathbf{e}_k}(\mathbf{u}). \tag{5}$$

This leads to a de Casteljau algorithm for computing the point $\mathbf{p}(\mathbf{t})$. We set $\mathbf{c}^0_{\mathbf{i}} = \mathbf{c}_{\mathbf{i}}$ for $|\mathbf{i}| = n$, and then for each $r = 1, \ldots, n$, let

$$\mathbf{c}^r_{\mathbf{i}} = \sum_{k=1}^{3} u_k \mathbf{c}^{r-1}_{\mathbf{i}+\mathbf{e}_k}, \qquad |\mathbf{i}| = n - r. \tag{6}$$

The last point computed in this algorithm is the point on the surface:

$$\mathbf{p}(\mathbf{t}) = \mathbf{c}^n_{0,0,0}.$$

Similarly to the univariate case, this follows by induction on r. We show more generally that

$$\mathbf{p}(t) = \sum_{|\mathbf{i}|=n-r} \mathbf{c}^r_{\mathbf{i}} B^{n-r}_{\mathbf{i}}(\mathbf{t}), \qquad r = 0, 1, \ldots, n. \tag{7}$$

Applying (5) to (7) gives

$$\begin{aligned} \mathbf{p}(\mathbf{t}) &= \sum_{|\mathbf{i}|=n-r} \mathbf{c}^r_{\mathbf{i}} \left(\sum_{k=1}^{3} u_k B^{n-r-1}_{\mathbf{i}-\mathbf{e}_k}(\mathbf{u}) \right) \\ &= \sum_{k=1}^{3} u_k \sum_{|\mathbf{i}|=n-r-1} \mathbf{c}^r_{\mathbf{i}+\mathbf{e}_k} B^{n-r-1}_{\mathbf{i}}(\mathbf{u}), \end{aligned}$$

which, by (6), is (7) with r replaced by $r+1$. The de Casteljau algorithm can be viewed as a tetrahedral scheme. The flow of computations in the case $n=2$ is as follows,

$$\begin{array}{ccccccccccccc} & & \mathbf{c}^0_{002} & & & & & & & & & & \\ & \mathbf{c}^0_{101} & & \mathbf{c}^0_{011} & & \rightarrow & & \mathbf{c}^1_{001} & & & \rightarrow & \mathbf{c}^2_{000} \\ \mathbf{c}^0_{200} & & \mathbf{c}^0_{110} & & \mathbf{c}^0_{020} & & \mathbf{c}^1_{100} & & \mathbf{c}^1_{010} & & & \end{array}$$

4 Derivatives

We would like to find the directional derivative of $\mathbf{p}$ in some vector direction $\mathbf{v} \in \mathbb{R}^2$, i.e.,

$$D_{\mathbf{v}}\mathbf{p}(\mathbf{t}) = \mathbf{v} \cdot \nabla \mathbf{p}(\mathbf{t}).$$

To do this we apply the chain rule. Observe that

$$D_{\mathbf{v}} B^n_{\mathbf{i}}(\mathbf{u}) = \sum_{k=1}^{3} \frac{\partial}{\partial u_k} B^n_{\mathbf{i}}(\mathbf{u}) D_{\mathbf{v}} u_k,$$

and by the definition of $B^n_{\mathbf{i}}(\mathbf{u})$, we easily see that

$$\frac{\partial}{\partial u_k} B^n_{\mathbf{i}}(\mathbf{u}) = n B^{n-1}_{\mathbf{i}-\mathbf{e}_k}(\mathbf{u}).$$

Therefore

$$\begin{aligned} D_{\mathbf{v}}\mathbf{p}(\mathbf{t}) &= n \sum_{|\mathbf{i}|=n} \mathbf{c}_{\mathbf{i}} \sum_{k=1}^{3} B^{n-1}_{\mathbf{i}-\mathbf{e}_k}(\mathbf{u}) D_{\mathbf{v}} u_k \\ &= n \sum_{|\mathbf{i}|=n-1} \mathbf{d}_{\mathbf{i}} B^{n-1}_{\mathbf{i}}(\mathbf{u}), \end{aligned}$$

where

$$\mathbf{d}_{\mathbf{i}} = \sum_{k=1}^{3} D_{\mathbf{v}} u_k \mathbf{c}_{\mathbf{i}+\mathbf{e}_k}.$$

One way of finding the directional derivatives $D_{\mathbf{v}} u_k$ is to take the directional derivative of equations (3–4) in the direction $\mathbf{v}$, which gives

$$\begin{aligned} D_{\mathbf{v}} u_1 + D_{\mathbf{v}} u_2 + D_{\mathbf{v}} u_3 &= 0, \\ D_{\mathbf{v}} u_1 \mathbf{a}_1 + D_{\mathbf{v}} u_2 \mathbf{a}_2 + D_{\mathbf{v}} u_3 \mathbf{a}_3 &= \mathbf{v}, \end{aligned}$$

which, like previously, can be solved using Cramer's rule. Notice that the three derivatives $D_{\mathbf{v}} u_k$, $k = 1, 2, 3$, are independent of $\mathbf{t}$, and depend only on $\mathbf{v}$ as well as the triangle vertices $\mathbf{a}_k$. We can therefore view them as coordinates of $\mathbf{v}$. They are similar to barycentric coordinates but they sum to zero instead of one. In conclusion, we have shown that

$$D_{\mathbf{v}} \mathbf{p}(\mathbf{t}) = n \sum_{|\mathbf{i}|=n-1} \Delta_{\mathbf{v}} \mathbf{c}_{\mathbf{i}} B_{\mathbf{i}}^{n-1}(\mathbf{u}), \tag{8}$$

where

$$\Delta_{\mathbf{v}} \mathbf{c}_{\mathbf{i}} = \sum_{k=1}^{3} v_k \mathbf{c}_{\mathbf{i}+\mathbf{e}_k},$$

and v_1, v_2, v_3 are the unique solutions to the equations

$$v_1 + v_2 + v_3 = 0, \tag{9}$$

$$v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + v_3 \mathbf{a}_3 = \mathbf{v}. \tag{10}$$

Continuing in this way we can also find higher order directional derivatives. For vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^2$,

$$D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_r} \mathbf{p}(\mathbf{t}) = \frac{n!}{(n-r)!} \sum_{|\mathbf{i}|=n-r} \Delta_{\mathbf{v}_1} \cdots \Delta_{\mathbf{v}_r} \mathbf{c}_{\mathbf{i}} B_{\mathbf{i}}^{n-r}(\mathbf{u}).$$

5 Joining surfaces smoothly

Consider now how we might build a piecewise polynomial surface from triangular Bézier surfaces. The main issue is how to fit together two triangular surfaces whose parameter domains share a common edge. Suppose then that $\mathbf{p}$ is the surface in (2) whose parameter domain is the triangle $T = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and that we denote by u_1, u_2, u_3 the barycentric coordinates of the point $\mathbf{t}$ with respect to T. Then, if $\mathbf{a}_4 \in \mathbb{R}^2$ is any point on the side of the edge $[\mathbf{a}_1, \mathbf{a}_2]$ opposite to $\mathbf{a}_3$, let $\tilde{\mathbf{p}}$ be a Bézier surface of the same degree n, defined on the triangle $U = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4]$,

$$\tilde{\mathbf{p}}(\mathbf{t}) = \sum_{|\mathbf{i}|=n} \tilde{\mathbf{c}}_{\mathbf{i}} B_{\mathbf{i}}^{n}(\tilde{\mathbf{u}}), \tag{11}$$

where $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_4)$ and the values $\tilde{u}_1, \tilde{u}_2, \tilde{u}_4$ are the barycentric coordinates of $\mathbf{t}$ with respect to U.

Suppose that the point $\mathbf{t}$ belongs to the common edge $[\mathbf{a}_1, \mathbf{a}_2]$. Then $u_k = \tilde{u}_k$, $k = 1, 2$, and $u_3 = \tilde{u}_4 = 0$, and

$$\mathbf{p}(\mathbf{t}) = \sum_{i+j=n} \mathbf{c}_{i,j,0} \frac{n!}{i!j!} u_1^i u_2^j,$$

and

$$\tilde{\mathbf{p}}(\mathbf{t}) = \sum_{i+j=n} \tilde{\mathbf{c}}_{i,j,0} \frac{n!}{i!j!} u_1^i u_2^j.$$

It follows that $\mathbf{p}$ and $\tilde{\mathbf{p}}$ join continuously on the common edge $[\mathbf{a}_1, \mathbf{a}_2]$ if and only if

$$\tilde{\mathbf{c}}_{i,j,0} = \mathbf{c}_{i,j,0}, \qquad i + j = n. \tag{12}$$

This is equivalent to saying that the control nets of $\mathbf{p}$ and $\tilde{\mathbf{p}}$ have the same boundary polygon on $[\mathbf{a}_1, \mathbf{a}_2]$. Under this condition we can define the continuous composite surface

$$\mathbf{s}(\mathbf{t}) := \begin{cases} \mathbf{p}(\mathbf{t}), & \mathbf{t} \in T; \\ \tilde{\mathbf{p}}(\mathbf{t}), & \mathbf{t} \in U. \end{cases}$$

Under what condition is $\mathbf{s}$ also C^1? Well, $\mathbf{s}$ is C^1 if its directional derivative $D_{\mathbf{v}}\mathbf{s}$ is continuous on $[\mathbf{a}_1, \mathbf{a}_2]$ for any vector $\mathbf{v}$ transversal to the vector $\mathbf{a}_2 - \mathbf{a}_1$, i.e., for any non-zero vector $\mathbf{v}$ that is not parallel to $\mathbf{a}_2 - \mathbf{a}_1$. Thus we compare the derivative formula for $\mathbf{p}$ in (8) with the corresponding formula for $\tilde{\mathbf{p}}$, namely,

$$D_{\mathbf{v}}\tilde{\mathbf{p}}(\mathbf{t}) = n \sum_{|\mathbf{i}|=n-1} \tilde{\Delta}_{\mathbf{v}} \tilde{\mathbf{c}}_i B_{\mathbf{i}}^{n-1}(\tilde{\mathbf{u}}),$$

where

$$\tilde{\Delta}_{\mathbf{v}} \tilde{\mathbf{c}}_{\mathbf{i}} = \tilde{v}_1 \tilde{\mathbf{c}}_{\mathbf{i}+\mathbf{e}_1} + \tilde{v}_2 \tilde{\mathbf{c}}_{\mathbf{i}+\mathbf{e}_2} + \tilde{v}_4 \tilde{\mathbf{c}}_{\mathbf{i}+\mathbf{e}_3},$$

and

$$\tilde{v}_1 + \tilde{v}_2 + \tilde{v}_4 = 0, \tag{13}$$

$$\tilde{v}_1 \mathbf{a}_1 + \tilde{v}_2 \mathbf{a}_2 + \tilde{v}_4 \mathbf{a}_4 = \mathbf{v}. \tag{14}$$

In the case that $\mathbf{t} \in [\mathbf{a}_1, \mathbf{a}_2]$ these formulas reduce to

$$D_{\mathbf{v}}\mathbf{p}(\mathbf{t}) = n \sum_{i+j=n-1} \Delta_{\mathbf{v}} \mathbf{c}_{i,j,0} \frac{n!}{i!j!} u_1^i u_2^j$$

and

$$D_{\mathbf{v}}\tilde{\mathbf{p}}(\mathbf{t}) = n \sum_{i+j=n-1} \tilde{\Delta}_{\mathbf{v}}\tilde{\mathbf{c}}_{i,j,0} \frac{n!}{i!j!} u_1^i u_2^j.$$

The condition for C^1 continuity is therefore

$$\tilde{\Delta}_{\mathbf{v}}\tilde{\mathbf{c}}_{i,j,0} = \Delta_{\mathbf{v}}\mathbf{c}_{i,j,0}, \qquad i+j=n-1,$$

which, due to (12), reduces to

$$\tilde{v}_4\tilde{\mathbf{c}}_{i,j,1} = (v_1-\tilde{v}_1)\mathbf{c}_{i+1,j,0} + (v_2-\tilde{v}_2)\mathbf{c}_{i,j+1,0} + v_3\mathbf{c}_{i,j,1}, \qquad i+j=n-1.$$

This can be rewritten in a form that is easy to remember:

$$\tilde{\mathbf{c}}_{i,j,1} = \mu_1\mathbf{c}_{i+1,j,0} + \mu_2\mathbf{c}_{i,j+1,0} + \mu_3\mathbf{c}_{i,j,1}, \qquad i+j=n-1,$$

where μ_1, μ_2, μ_3 are the barycentric coordinates of the point $\mathbf{a}_4$ with respect to T,

$$\mu_1+\mu_2+\mu_3 = 1,$$
$$\mu_1\mathbf{a}_1+\mu_2\mathbf{a}_2+\mu_3\mathbf{a}_3 = \mathbf{a}_4.$$

This follows from subtracting (14) from (10). The coordinates μ_1 and μ_2 are positive but μ_3 is negative.