Spline curves

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In this lecture we introduce spline curves and study some of their basic properties.

1 Spline curves

For any integers $d \geq 0$ and $n \geq 1$, we call a sequence $\mathbf{t} = (t_1, t_2, \ldots, t_{n+d+1})$, $t_i \in \mathbb{R}$, a *knot vector* if $t_i \leq t_{i+1}$ and $t_i < t_{i+d+1}$. Such a sequence of knots together with a sequence of *control points* $\mathbf{c}_i \in \mathbb{R}^m$, $i = 1, \ldots, n$, define a *spline curve*

$$\mathbf{s}(t) = \sum_{i=1}^{n} \mathbf{c}_i N_i^d(t), \qquad t \in \mathbb{R}, \tag{1}$$

where the functions N_i^d are *B-splines*. These B-splines can be defined recursively:

$$N_i^0(t) = \begin{cases} 1 & t \in [t_i, t_{i+1}); \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

and for $d \geq 1$,

$$N_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i} N_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1}^{d-1}(t). \tag{3}$$

We use the convention here that

$$N_i^{r-1} = \frac{N_i^{r-1}}{t_{i+r} - t_i} = 0, \qquad \text{if } t_{i+r} = t_i.$$

From this recursion it follows that N_i^d is a piecewise polynomial of degree d, which is positive in (t_i, t_{i+d+1}) and zero outside $[t_i, t_{i+d+1}]$. In practice we restrict $\mathbf{s}$ to being the parametric curve $\mathbf{s} : [t_{d+1}, t_{n+1}) \to \mathbb{R}^m$. The parameter domain $[t_{d+1}, t_{n+1})$ consists of the knot intervals $[t_k, t_{k+1})$, $k = d+1, \ldots, n$. In the knot interval $[t_k, t_{k+1})$, the spline $\mathbf{s}$ depends on, and only on, the $d+1$ control points $\mathbf{c}_{k-d}, \ldots, \mathbf{c}_k$.

2 Evaluation

Similar to Bézier curves, there are two ways of evaluating a spline curve. One way is to use the recursion (3) and then the formula (1). Suppose $t \in [t_k, t_{k+1})$ for some $k \in \{d+1, \ldots, n\}$. Then,

$$\mathbf{s}(t) = \sum_{i=k-d}^{k} \mathbf{c}_i N_i^d(t),$$

and we only need to compute $N_{k-d}^d(t), \ldots, N_k^d(t)$, for all the other B-splines are zero in $[t_k, t_{k+1})$. The recursion (3) can then be carried out in a triangular scheme,

$$\begin{array}{ccccccc} 1 & = & N_k^0 & N_{k-1}^1 & N_{k-2}^2 & \cdots & N_{k-d}^d \\ & & & N_k^1 & N_{k-1}^2 & \cdots & N_{k-d+1}^d \\ & & & & N_k^2 & \cdots & N_{k-d+2}^d \\ & & & & & \ddots & \vdots \\ & & & & & & N_k^d \end{array}$$

Alternatively, one can use a more direct recursion algorithm. Let $\mathbf{c}_i^0 = \mathbf{c}_i$, $i = k-d, \ldots, k$. Then for $r = 1, \ldots, d$, and $i = k-d+r, \ldots, k$, let

$$\mathbf{c}_i^r = \frac{t_{i+d+1-r} - t}{t_{i+d+1-r} - t_i} \mathbf{c}_{i-1}^{r-1} + \frac{t - t_i}{t_{i+d+1-r} - t_i} \mathbf{c}_i^{r-1}. \tag{4}$$

One can show that the last point computed is $\mathbf{c}_k^d = \mathbf{s}(t)$. Similar to the de Casteljau algorithm, this can be shown by showing, more generally, by induction on r, that

$$\mathbf{s}(t) = \sum_{i=k-d+r}^{k} \mathbf{c}_i^r N_i^{d-r}(t). \tag{5}$$

This algorithm can also be arranged in a triangular scheme, here row-wise,

$$
\begin{array}{ccccccccc}
\mathbf{c}_{k-d}^0 & & \mathbf{c}_{k-d+1}^0 & & \mathbf{c}_{k-d+2}^0 & & \cdots & & \mathbf{c}_k^0 \\
& \mathbf{c}_{k-d+1}^1 & & \mathbf{c}_{k-d+2}^1 & & \cdots & & \mathbf{c}_k^1 & \\
& & \ddots & & & & \iddots & & \\
& & & \mathbf{c}_{k-1}^{d-1} & & \mathbf{c}_k^{d-1} & & & \\
& & & & \mathbf{c}_k^d & & & &
\end{array}
$$

3 Polar forms

In analogy to Bézier curves we can express the control points of a spline curve in terms of polar forms. There is a polar form for each polynomial piece of $\mathbf{s}$. Recall that the d-variate polar form $\mathcal{P}[p](x_1, \ldots, x_d)$ of the polynomial

$$p(x) = \sum_{i=0}^{d} a_i x^i, \qquad a_i \in \mathbb{R}, \tag{6}$$

is

$$\mathcal{P}[p](x_1, \ldots, x_d) = \sum_{i=0}^{d} a_i S_i(x_1, \ldots, x_d),$$

where S_i is the symmetric polynomial

$$S_i(x_1, \ldots, x_d) = \sum_{1 \le k_1 < k_2 < \cdots < k_i \le d} x_{k_1} x_{k_2} \cdots x_{k_i} \Big/ \binom{d}{i}. \tag{7}$$

Consider again the spline curve $\mathbf{s}$ restricted to some non-empty interval $[t_k, t_{k+1})$, $d+1 \le k \le n$. In this interval $\mathbf{s}$ is a polynomial which we can denote by $\mathbf{s}_k$,

$$\mathbf{s}_k(t) = \sum_{i=k-d}^{k} \mathbf{c}_i N_i^d(t), \qquad t \in [t_k, t_{k+1}). \tag{8}$$

Theorem 1 *For $i = k-d, \ldots, k$,*

$$\mathbf{c}_i = \mathcal{P}[\mathbf{s}_k](t_{i+1}, \ldots, t_{i+d}).$$

Proof. To prove this let

$$\mathbf{c}_i^r = \mathcal{P}[\mathbf{s}_k](t_{i+1}, \ldots, t_{i+d-r}, \underbrace{t, \ldots, t}_{r}).$$

Since $\mathcal{P}[\mathbf{s}_k]$ is multi-affine and symmetric, and since

$$t = (1 - \alpha)t_i + \alpha t_{i+d-r+1},$$

where

$$\alpha = \frac{t - t_i}{t_{i+d-r+1} - t_i},$$

it follows that $\mathbf{c}_i^r$ satisfies the recursion (4). Therefore,

$$\mathbf{c}_k^d = \sum_{i=k-d}^{k} \mathbf{c}_i^0 N_i^d(t),$$

and so, by the diagonal property of $\mathcal{P}[\mathbf{s}_k]$,

$$\mathbf{s}_k(t) = \sum_{i=k-d}^{k} \mathbf{c}_i^0 N_i^d(t) = \sum_{i=k-d}^{k} \mathcal{P}[\mathbf{s}_k](t_{i+1}, \ldots, t_{i+d}) N_i^d(t). \tag{9}$$

This equation shows that any polynomial of degree $\leq d$ in the interval $[t_k, t_{k+1})$ can be expressed as a linear combination of $N_{k-d}^d, \ldots, N_k^d$, and since there are $d + 1$ of these, they must be linearly independent. Thus we can equate the coefficients $\mathbf{c}_i$ in (8) with those in (9). □

4 Continuity

Consider now the continuity of the spline $\mathbf{s}$. If a knot z in the knot vector $\mathbf{t}$ occurs r times we say that z is an *r-fold* knot, or that z has *multiplicity* r.

Lemma 1 *The spline $\mathbf{s}$, of degree $d \geq 1$, is continuous at a knot of multiplicity d.*

Proof. We may assume that for some i,

$$z = t_{i+1} = \cdots = t_{i+d},$$

and

$$t_i < z < t_{i+d+1}.$$

Then $\mathbf{s}|_{[t_i,z)} = \mathbf{s}_i$ and $\mathbf{s}|_{[z,t_{i+d+1})} = \mathbf{s}_{i+d}$ and so the two polynomial pieces $\mathbf{s}_i$ and $\mathbf{s}_{i+d}$ are adjacent and the task is to show that they are equal at z. Using the polar forms of $\mathbf{s}_i$ and $\mathbf{s}_{i+d}$ we find

$$\mathbf{s}_i(z) = \mathcal{P}[\mathbf{s}_i](\underbrace{z,\ldots,z}_{d}) = \mathcal{P}[\mathbf{s}_i](t_{i+1},\ldots,t_{i+d}) = \mathbf{c}_i,$$

and

$$\mathbf{s}_{i+d}(z) = \mathcal{P}[\mathbf{s}_{i+d}](\underbrace{z,\ldots,z}_{d}) = \mathcal{P}[\mathbf{s}_{i+d}](t_{i+1},\ldots,t_{i+d}) = \mathbf{c}_i,$$

and so both pieces are equal to $\mathbf{c}_i$ at z. □

Lemma 2 *A spline* $\mathbf{s}$ *of degree* d *is continuous at a knot* z *of multiplicity* r *for any* $r \leq d$.

Proof. Since any spline of degree r, on the same knot vector as $\mathbf{s}$, is continuous at z, the B-splines N_i^r (on the same knot vector) are also continuous at z. So, by the recursion formula for B-splines (3), the B-splines N_i^{r+1} are also continuous at z, and similarly for N_i^{r+2}, and so on. Thus the B-splines N_i^d are also continuous at z and so too is $\mathbf{s}$. □

5 Derivatives

Using polar forms, we next find a formula for the first derivative of the spline $\mathbf{s}$ as a spline of degree $d-1$.

Lemma 3 *Let* p *be the polynomial of degree* $\leq d$ *in (6) and* $\mathcal{P}[p]$ *its polar form (7). For any* $a, b \in \mathbb{R}$ *with* $a \neq b$,

$$\mathcal{P}[p'](x_1,\ldots,x_{d-1}) = d\frac{\mathcal{P}[p](x_1,\ldots,x_{d-1},b) - \mathcal{P}[p](x_1,\ldots,x_{d-1},a)}{b-a}. \quad (10)$$

Proof. By the definition of S_i,

$$S_i(x_1,\ldots,x_d) = \frac{d-i}{d}S_i(x_1,\ldots,x_{d-1}) + \frac{i}{d}x_d S_{i-1}(x_1,\ldots,x_{d-1}),$$

and therefore, for any a, b,

$$S_i(x_1, \ldots, x_{d-1}, b) - S_i(x_1, \ldots, x_{d-1}, a) = \frac{i}{d}(b-a)S_{i-1}(x_1, \ldots, x_{d-1}).$$

Since the derivative of p is

$$p'(x) = \sum_{i=1}^{d} i a_i x^{i-1}, \tag{11}$$

this implies that if $a \neq b$,

$$\begin{aligned} \mathcal{P}[p'](x_1, \ldots, x_{d-1}) &= \sum_{i=1}^{d} i a_i S_{i-1}(x_1, \ldots, x_{d-1}) \\ &= \frac{d}{b-a} \sum_{i=1}^{d} a_i \big(S_i(x_1, \ldots, x_{d-1}, b) - S_i(x_1, \ldots, x_{d-1}, b)\big). \end{aligned}$$

Since $S_0 = 1$ this sum can be extended to include $i = 0$, and then we obtain (10). □

Lemma 4

$$\mathbf{s}'(t) = \sum_{i=2}^{n} \mathbf{d}_i N_i^{d-1}(t),$$

where

$$\mathbf{d}_i = \frac{d}{t_{i+d} - t_i}(\mathbf{c}_i - \mathbf{c}_{i-1}).$$

Proof. Consider again one of the spline segments $\mathbf{s}_k$. Since its deriavtive is a polynomial of degree $\leq d-1$, there must be coefficients $\mathbf{d}_{k-d+1}, \ldots, \mathbf{d}_k$ such that

$$\mathbf{s}_k'(t) = \sum_{i=k-d+1}^{k} \mathbf{d}_i N_i^{d-1}(t).$$

From the polar form of $\mathbf{s}_k'$,

$$\mathbf{d}_i = \mathcal{P}[\mathbf{s}_k'](t_{i+1}, \ldots, t_{i+d-1}).$$

Using the formula of the previous lemma with $a = t_i$ and $b = t_{i+d}$, it follows that

$$\mathbf{d}_i = \frac{d}{t_{i+d} - t_i}(\mathbf{c}_i - \mathbf{c}_{i-1}).$$

Since these coefficients are independent of k, the result follows. □

Thus $\mathbf{s}'$ is a spline of degree $d-1$ on the same knot vector as $\mathbf{s}$. We can continue to differentiate in this way, and thus express the k-th derivative $\mathbf{s}^{(k)}$ as a spline of degree $d-k$ on the same knot vector.

6 Smoothness

We can now establish the smoothness of a spline.

Theorem 2 *A spline* $\mathbf{s}$ *of degree* d *has order of continuity* C^{d-r} *at a knot* z *of multiplicity* r *for any* $r \leq d$.

Proof. Let $k \in \{0, 1, \ldots, d-r\}$ and consider the k-th derivative $\mathbf{s}^{(k)}$. By the differentiation formula of the previous section, $\mathbf{s}^{(k)}$ is itself a spline, of degree $d-k$, on the same knot vector as $\mathbf{s}$. Thus, since $d-k \geq r$, Lemma 2 implies that $\mathbf{s}^{(k)}$ is continuous at z. Thus the derivatives of $\mathbf{s}$ of orders $0, 1, \ldots, d-r$ are continuous at z. □

In particular, at a simple knot, i.e., a knot with multiplicity 1, $\mathbf{s}$ has smoothness C^{d-1}.