Surface subdivision

Michael S. Floater

September 30, 2011

In this lecture we recall the schemes for subdivision of B-spline curves and describe corresponding schemes for tensor-product spline surfaces defined by rectangular meshes of control points. We then extend some of these to irregular control meshes, leading to the Catmull-Clark and Doo-Sabin schemes. We also describe the Loop subdivision scheme for generating a surface from a triangular control mesh.

1 Curve subdivision

Recall that a *subdivision curve* is a curve generated by iterative refinement of a given polygon, called the *control polygon*. The limit curve can be rendered by simply rendering the polygon resulting from sufficiently many refinements. Both Bezier curves and spline curves are examples of subdivision curves. Consider, for example, Chaikin's scheme, which generates a C^1 quadratic spline curve (with uniform knots). We begin with a control polygon $\ldots, \mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}, \ldots$, as in Figure 1(a). We then generate a refined polygon by the rule

$$\begin{aligned} \mathbf{v}^1_{2i} &= \frac{3}{4}\mathbf{v}_{i-1} + \frac{1}{4}\mathbf{v}_i, \\ \mathbf{v}^1_{2i+1} &= \frac{1}{4}\mathbf{v}_{i-1} + \frac{3}{4}\mathbf{v}_i, \end{aligned}$$

as shown in in Figure 1(b).

The full subdivision scheme is as follows.

1. Set $\mathbf{v}^0_i = \mathbf{v}_i$, for all $i \in \mathbb{Z}$.

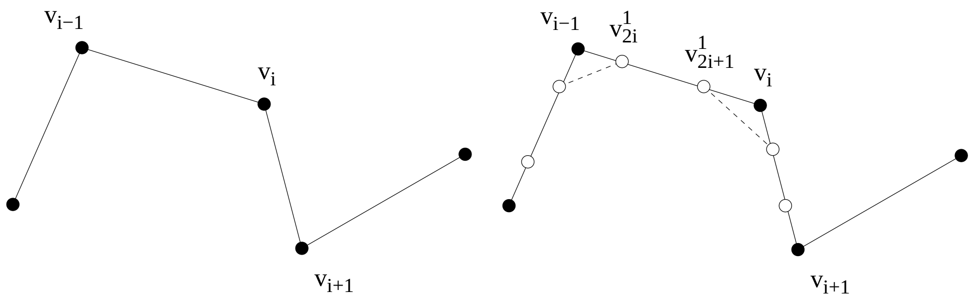

Figure 1: Left: (a) initial polygon. Right: (b) the first refinement.

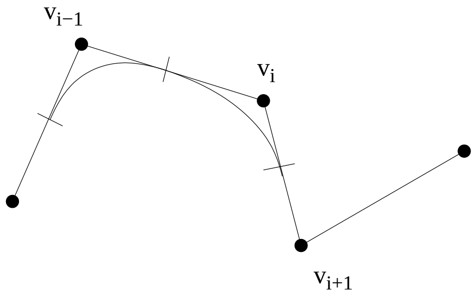

Figure 2: The limit curve.

2. For $n = 1, 2, \ldots$, set

$$\mathbf{v}_{2i}^{n} = \frac{3}{4}\mathbf{v}_{i-1}^{n-1} + \frac{1}{4}\mathbf{v}_{i}^{n-1},$$
$$\mathbf{v}_{2i+1}^{n} = \frac{1}{4}\mathbf{v}_{i-1}^{n-1} + \frac{3}{4}\mathbf{v}_{i}^{n-1}.$$

The number of points doubles at each iteration. The limiting curve is shown in Figure 2.

The general (linear) subdivision scheme is

$$\mathbf{v}_{i}^{n} = \sum_{k\in\mathbb{Z}} a_{i-2k}\mathbf{v}_{k}^{n-1},$$

where $a_0, a_1, \ldots, a_m$ is the (finite) *subdivision mask* (all other a_i are zero). The mask for Chaikin's scheme is

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

The mask can be split into two masks, one for even and one for odd indices,

$$\begin{aligned}
\mathbf{v}_{2i}^n &= \sum_{k\in\mathbb{Z}} a_{2k}\mathbf{v}_{i-k}^{n-1}, \\
\mathbf{v}_{2i+1}^n &= \sum_{k\in\mathbb{Z}} a_{2k+1}\mathbf{v}_{i-k}^{n-1}.
\end{aligned}$$

In Chaikin's scheme, these equations become

$$\begin{aligned}
\mathbf{v}_{2i}^n &= a_0\mathbf{v}_i^{n-1} + a_2\mathbf{v}_{i-1}^{n-1} = \frac{1}{4}\mathbf{v}_i^{n-1} + \frac{3}{4}\mathbf{v}_{i-1}^{n-1}, \\
\mathbf{v}_{2i+1}^n &= a_1\mathbf{v}_i^{n-1} + a_3\mathbf{v}_{i-1}^{n-1} = \frac{3}{4}\mathbf{v}_i^{n-1} + \frac{1}{4}\mathbf{v}_{i-1}^{n-1},
\end{aligned}$$

and the two masks are

$$\begin{pmatrix} a_0 & a_2 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{4} & \tfrac{3}{4} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_1 & a_3 \end{pmatrix} = \begin{pmatrix} \tfrac{3}{4} & \tfrac{1}{4} \end{pmatrix}.$$

Another example is a C^2 cubic spline curve (again with uniform knots). The mask is

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

If we split into the two masks (a_0, a_2, a_4) and (a_1, a_3), we get the scheme

$$\begin{aligned}
\mathbf{v}_{2i}^n &= \frac{1}{8}(\mathbf{v}_i^{n-1} + 6\mathbf{v}_{i-1}^{n-1} + \mathbf{v}_{i-2}^{n-1}), \\
\mathbf{v}_{2i+1}^n &= \frac{1}{2}(\mathbf{v}_i^{n-1} + \mathbf{v}_{i-1}^{n-1}),
\end{aligned}$$

see Figure 3. A uniform C^{d-1} spline curve of degree d can be generated by the mask

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{d+1} \end{pmatrix} = \frac{1}{2^d}\begin{pmatrix} \binom{d+1}{0} & \binom{d+1}{1} & \cdots & \binom{d+1}{d+1} \end{pmatrix}.$$

2 Surface subdivision

We can generate a tensor-product spline surface from its rectangular mesh of control points by repeatedly subdividing first in one direction and then in

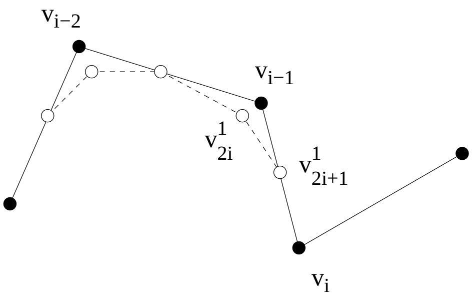

Figure 3: One refinement of cubic subdivision.

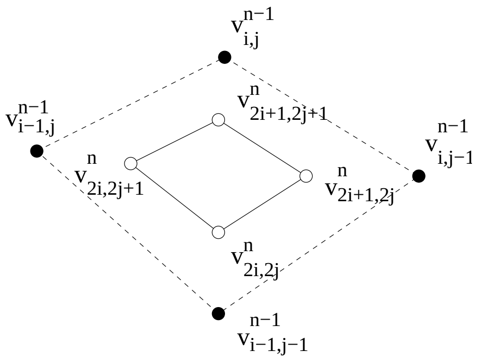

Figure 4: Biquadratic subdivision step.

the other. For example, for C^1 biquadratic surfaces the resulting scheme is

$$\begin{aligned}
\mathbf{v}^n_{2i,2j} &= \frac{1}{16}(9\mathbf{v}^{n-1}_{i-1,j-1} + 3\mathbf{v}^{n-1}_{i,j-1} + 3\mathbf{v}^{n-1}_{i-1,j} + \mathbf{v}^{n-1}_{i,j}) \\
\mathbf{v}^n_{2i+1,2j} &= \frac{1}{16}(3\mathbf{v}^{n-1}_{i-1,j-1} + 9\mathbf{v}^{n-1}_{i,j-1} + \mathbf{v}^{n-1}_{i-1,j} + 3\mathbf{v}^{n-1}_{i,j}) \\
\mathbf{v}^n_{2i,2j+1} &= \frac{1}{16}(3\mathbf{v}^{n-1}_{i-1,j-1} + \mathbf{v}^{n-1}_{i,j-1} + 9\mathbf{v}^{n-1}_{i-1,j} + 3\mathbf{v}^{n-1}_{i,j}) \\
\mathbf{v}^n_{2i+1,2j+1} &= \frac{1}{16}(\mathbf{v}^{n-1}_{i-1,j-1} + 3\mathbf{v}^{n-1}_{i,j-1} + 3\mathbf{v}^{n-1}_{i-1,j} + 9\mathbf{v}^{n-1}_{i,j}),
\end{aligned}$$

see Figure 4.

There are four submasks

$$\frac{1}{16}\begin{pmatrix} 3 & 1 \\ 9 & 3 \end{pmatrix}, \quad \frac{1}{16}\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}, \quad \frac{1}{16}\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}, \quad \frac{1}{16}\begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix}.$$

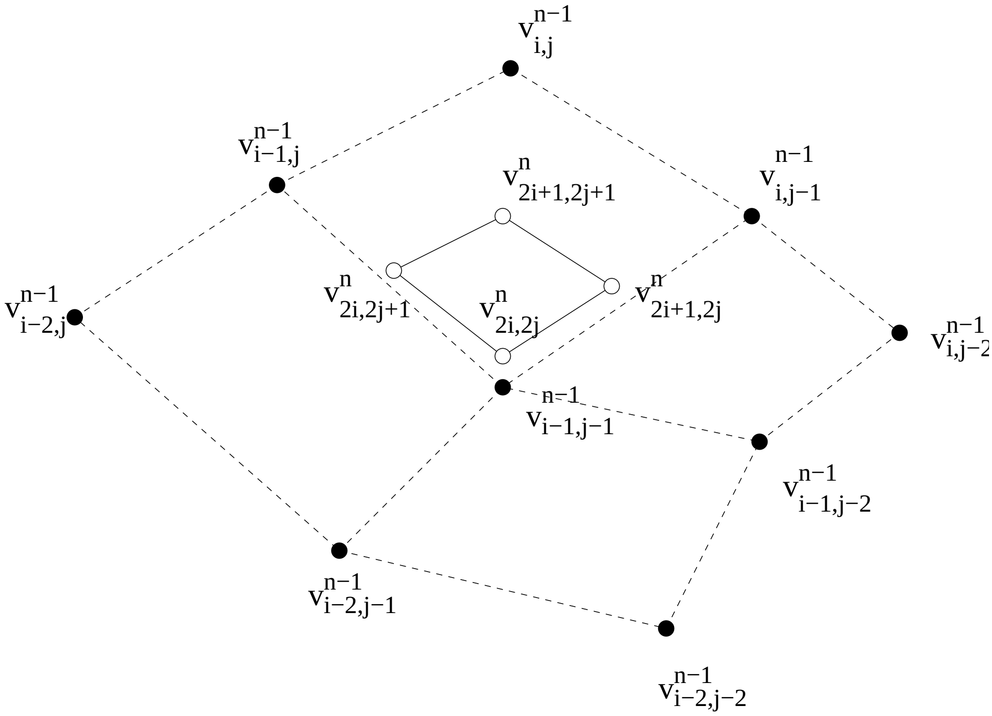

Figure 5: Bicubic subdivision step.

They are tensor-products of the quadratic curve masks. For example

$$\frac{1}{16}\begin{pmatrix}3 & 1\\ 9 & 3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1\\ 3\end{pmatrix}\frac{1}{4}\begin{pmatrix}3 & 1\end{pmatrix}.$$

Consider next bicubic spline surfaces. The mask for cubic curves is

$$\begin{pmatrix}a_0 & a_1 & a_2 & a_3 & a_4\end{pmatrix} = \frac{1}{8}\begin{pmatrix}1 & 4 & 6 & 4 & 1\end{pmatrix}.$$

and the two submasks are

$$\frac{1}{8}\begin{pmatrix}1 & 6 & 1\end{pmatrix} \qquad \text{and} \qquad \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}.$$

If we take tensor-products of these two submasks we get the four bicubic masks

$$\underbrace{\frac{1}{64}\begin{pmatrix}1 & 6 & 1\\ 6 & 36 & 6\\ 1 & 6 & 1\end{pmatrix}}_{\text{Mask A}}, \quad \underbrace{\frac{1}{16}\begin{pmatrix}1 & 1\\ 6 & 6\\ 1 & 1\end{pmatrix}, \quad \frac{1}{16}\begin{pmatrix}1 & 6 & 1\\ 1 & 6 & 1\end{pmatrix}}_{\text{Masks B}}, \quad \underbrace{\frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}}_{\text{Mask C}}.$$

These are used to compute the four new points

$$\begin{pmatrix}\mathbf{v}^n_{2i+1,2j} & \mathbf{v}^n_{2i+1,2j+1}\\ \mathbf{v}^n_{2i,2j} & \mathbf{v}^n_{2i,2j+1}\end{pmatrix}$$

from the old points

$$\begin{pmatrix}\mathbf{v}^{n-1}_{i-2,j} & \mathbf{v}^{n-1}_{i-1,j} & \mathbf{v}^{n-1}_{i,j}\\ \mathbf{v}^{n-1}_{i-2,j-1} & \mathbf{v}^{n-1}_{i-1,j-1} & \mathbf{v}^{n-1}_{i,j-1}\\ \mathbf{v}^{n-1}_{i-2,j-2} & \mathbf{v}^{n-1}_{i-1,j-2} & \mathbf{v}^{n-1}_{i,j-2}\end{pmatrix}.$$

3 Catmull-Clark subdivision

Catmull-Clark subdivision is a generalization of the C^2 **bicubic** scheme for rectangular meshes to arbitrary quadrilateral meshes. The limit surface is C^2 except at extraordinary points. It is enough to define the masks associated with Figure 7. In the figure, 5 faces meet at the vertex $\mathbf{v}^{n-1}$. In general there will be N faces. In the 'canonical' case we have $N = 4$. As for the $N = 4$ bicubic case, there are three types of points: vertex points $\mathbf{v}$, edge points $\mathbf{e}$, and face points $\mathbf{f}$, and there are three associated masks. The algorithm goes in three steps.

Step 1. Compute the new face points. We use Mask C as before:

$$\mathbf{f}^n_i = \frac{1}{4}(\mathbf{v}^{n-1} + \mathbf{e}^{n-1}_i + \mathbf{e}^{n-1}_{i+1} + \mathbf{f}^{n-1}_i).$$

Step 2. Compute the new edge points. We use Mask B as before:

$$\mathbf{e}^n_i = \frac{1}{16}(\mathbf{e}^{n-1}_{i-1} + \mathbf{f}^{n-1}_{i-1} + 6\mathbf{v}^{n-1} + 6\mathbf{e}^{n-1}_i + \mathbf{e}^{n-1}_{i+1} + \mathbf{f}^{n-1}_i).$$

Using the new face points computed in the first step, this computation reduces to

$$\mathbf{e}^n_i = \frac{1}{4}(\mathbf{v}^{n-1} + \mathbf{e}^{n-1}_i + \mathbf{f}^n_{i-1} + \mathbf{f}^n_i).$$

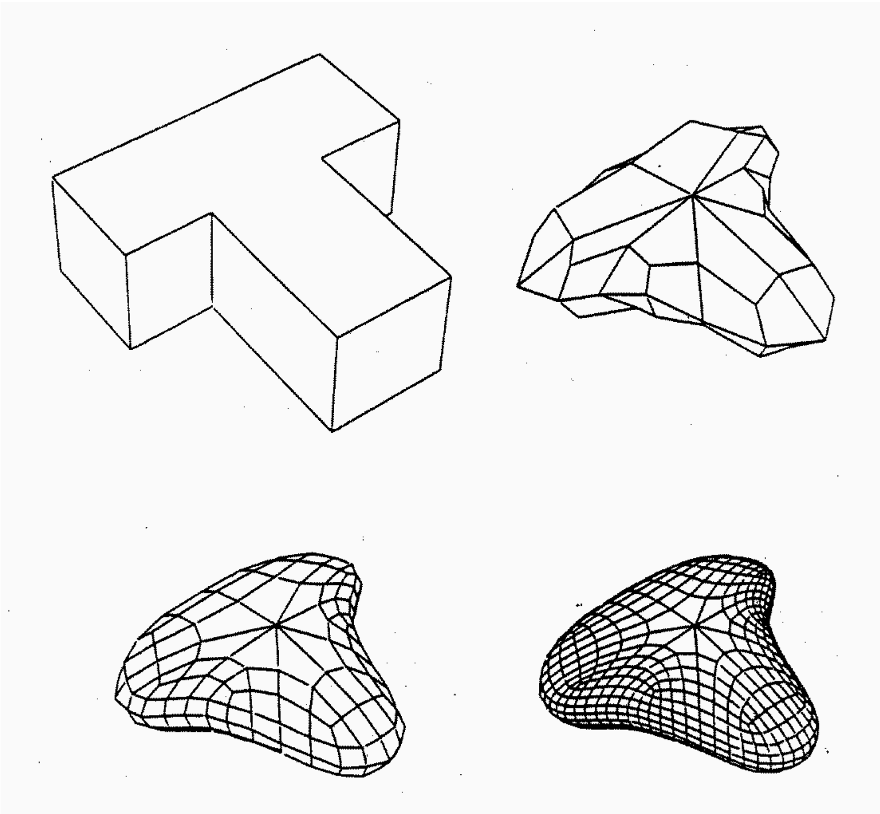

Figure 6: Catmull-Clark scheme.

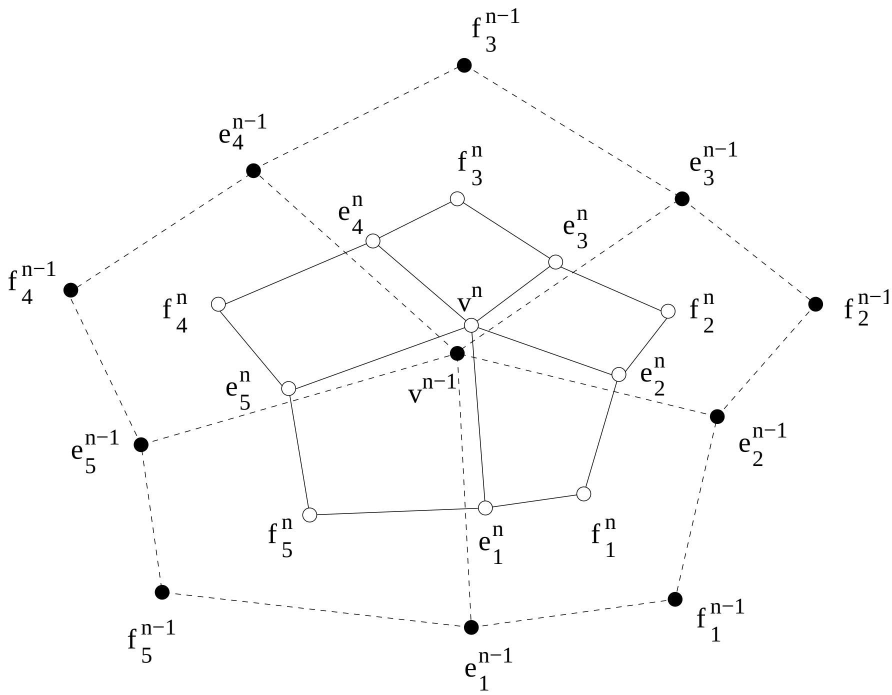

Figure 7: Vertices in the Catmull-Clark scheme.

Step 3. Compute the new vertex point. For $N = 4$ the rule for Mask A is

$$\mathbf{v}^n = \frac{1}{64}\Big(36\mathbf{v}^{n-1} + 6\sum_{i=1}^{4}\mathbf{e}_i^{n-1} + \sum_{i=1}^{4}\mathbf{f}_i^{n-1}\Big),$$

which can be expressed as

$$\mathbf{v}^n = \frac{1}{4}\Big(2\mathbf{v}^{n-1} + \frac{1}{4}\sum_{i=1}^{4}\mathbf{e}_i^{n-1} + \frac{1}{4}\sum_{i=1}^{4}\mathbf{f}_i^{n}\Big).$$

Catmull and Clark proposed the generalization

$$\mathbf{v}^n = \frac{1}{N}\Big((N-2)\mathbf{v}^{n-1} + \frac{1}{N}\sum_{i=1}^{N}\mathbf{e}_i^{n-1} + \frac{1}{N}\sum_{i=1}^{N}\mathbf{f}_i^{n}\Big).$$

This formula ensures C^1 continuity at the extraordinary points. It can be shown that C^2 continuity at extraordinary points is impossible without using larger masks.

4 Doo-Sabin subdivision

Doo-Sabin subdivision surfaces generalize C^1 biquadratic spline surfaces to arbitrary quadrilateral meshes. Tangent plane (C^1) continuity is again achieved at the extrordinary points. We will not give the details, just illustrate with Figure 8.

5 Loop subdivision

This is a subdivision scheme for arbitrary triangular meshes, based on so-called 'box-splines' (which is beyond the scope of this course), specifically C^2 quartic box-splines; see Figure 9. In this scheme we only compute vertex points and edge points, so there are only two masks. In each subdivision step, each triangle is replaced by four triangles: a so-called **1-4 split**; see Figure 10. Suppose we have the situation of Figure 11. Here, the number of neighbouring triangles is $N = 5$. The 'canonical' case is $N = 6$ in which case the scheme reduces to 'box-spline' subdivision, yielding a C^2 surface. The algorithm has just two steps.

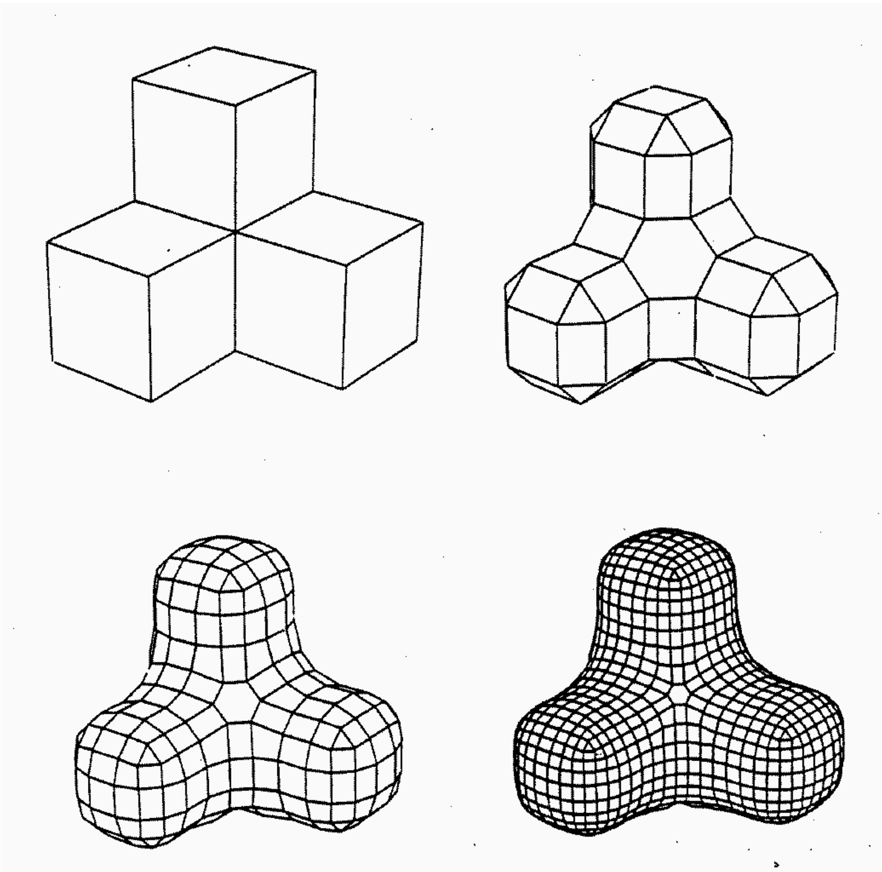

Figure 8: Doo-Sabin scheme.

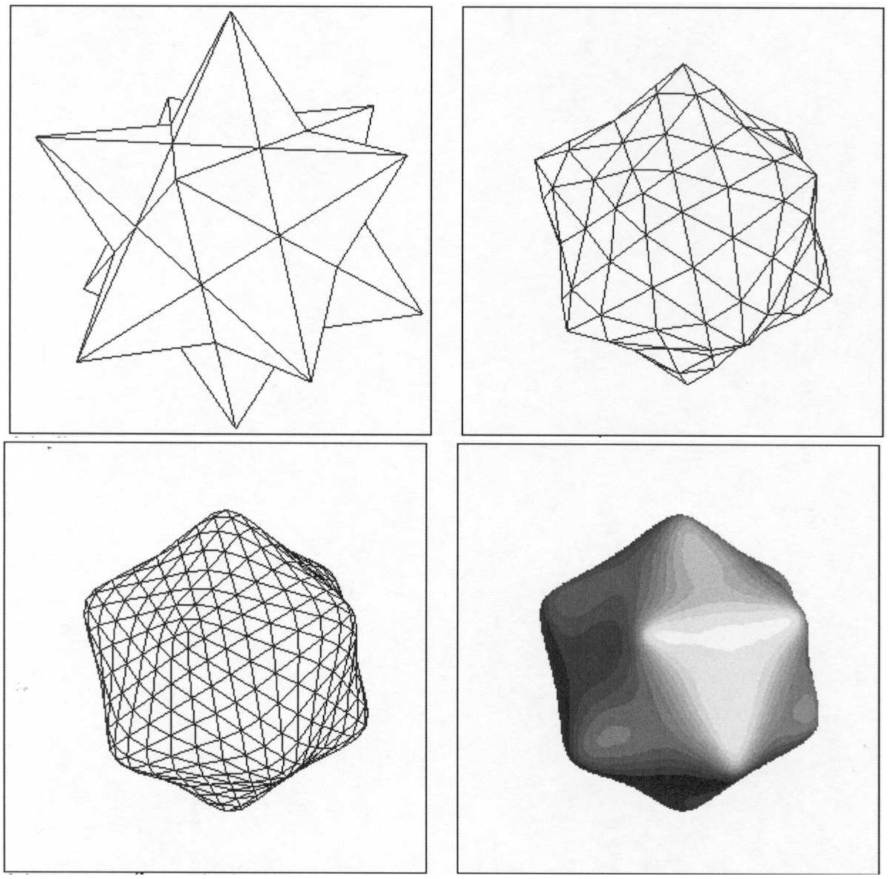

Figure 9: Loop scheme.

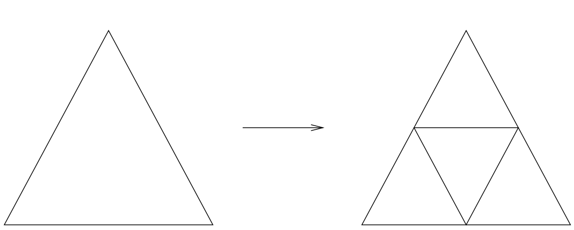

Figure 10: A 1-4 split.

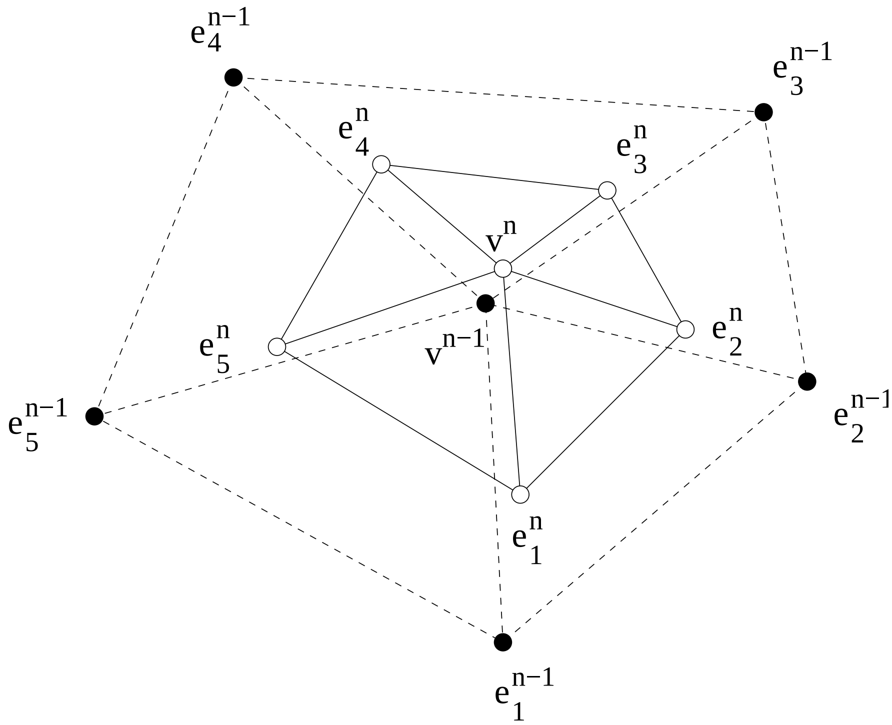

Figure 11: Vertices in the Loop scheme.

Step 1. Compute the new edge points by the rule

$$\mathbf{e}_i^n = \frac{1}{8}(3\mathbf{v}^{n-1} + 3\mathbf{e}_i^{n-1} + \mathbf{e}_{i-1}^{n-1} + \mathbf{e}_{i+1}^{n-1}).$$

Step 2. Compute the new vertex points. The rule for 'box-splines' in the case $N = 6$ is

$$\mathbf{v}^n = \frac{5}{8}\mathbf{v}^{n-1} + \frac{3}{8}\Big(\frac{1}{6}\sum_{i=1}^{6}\mathbf{e}_i^{n-1}\Big).$$

Loop proposed the formula

$$\mathbf{v}^n = \alpha_N \mathbf{v}^{n-1} + (1-\alpha_N)\Big(\frac{1}{N}\sum_{i=1}^{N}\mathbf{e}_i^{n-1}\Big)$$

for general N, and showed that with the weighting

$$\alpha_N = \Big(\frac{3}{8} + \frac{1}{4}\cos(2\pi/N)\Big)^2 + \frac{3}{8},$$

the limit surface is C^1 at the extraordinary points. The surface is a generalization of a box-spline surface because $\alpha_6 = \frac{5}{8}$.