Oblig 3 - exercise 4.5

April 24, 2014

4.5 Prove Lemma 4.2 in the general case where $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are not p+1-regular. Hint: Augment both $\boldsymbol{\tau}$ and $\boldsymbol{t}$ by inserting p + 1 identical knots at the beginning and end.

Proof. Let $\boldsymbol{\tau} = (\tau_i)_{i=1}^{n+p+1}$ and $\boldsymbol{t} = (t_i)_{i=1}^{m+p+1}$ with $\boldsymbol{\tau} \subseteq \boldsymbol{t}$. We follow the hint and insert p+1 knots into $\boldsymbol{\tau}$ at a position $a < t_1$ and p+1 knots at a position $b > t_{m+p+1}$. Denote this knot vector $\boldsymbol{\tau}'$. We also insert the same knots into $\boldsymbol{t}$ to form $\boldsymbol{t}'$.

Adding p+1 knots at both ends extends the corresponding splinespaces by adding p+1 B-splines at both ends. It is easy to see that $\mathbb{S}_{\boldsymbol{\tau}} \subseteq \mathbb{S}_{\boldsymbol{\tau}'}$ since a $f \in \mathbb{S}_{\boldsymbol{\tau}}$ can be represented in $\mathbb{S}_{\boldsymbol{\tau}'}$ by adding p+1 zero–coefficients in both ends. We also know that $\mathbb{S}_{\boldsymbol{\tau}'} \subseteq \mathbb{S}_{\boldsymbol{t}'}$ by Lemma 4.2 (since $\boldsymbol{\tau}'$ and $\boldsymbol{t}'$ are p+1-regular) and hence $\mathbb{S}_{\boldsymbol{\tau}} \subseteq \mathbb{S}_{\boldsymbol{\tau}'} \subseteq \mathbb{S}_{\boldsymbol{t}'}$. Moreover we know that $\mathbb{S}_{\boldsymbol{t}}$ is the subspace of $\mathbb{S}_{\boldsymbol{t}'}$ consisting of splines with p+1 zero-coefficients at both ends.

Let us now take $f \in \mathbb{S}_{\boldsymbol{\tau}}$ and show that $f \in \mathbb{S}_{\boldsymbol{t}}$. It is easy to see that $f \in \mathbb{S}_{\boldsymbol{t}'}$. Since $f(x) = 0$ in the intervals $[a, \tau_1]$ and $[\tau_{n+p+1}, b]$, the coefficients of f wrt $\mathbb{S}_{t'}$ must have at least $p + 1$ zeros at either end. Therefore $f \in \mathbb{S}_{\boldsymbol{t}}$. Since a, b and f was arbitrary we conclude that $\mathbb{S}_{\boldsymbol{\tau}} \subseteq \mathbb{S}_{\boldsymbol{t}}$.

□