Deadline: Thursday 19. October 2017

MANDATORY ASSIGNMENT MAT-INF4300/9300—FALL 2017

INFORMATION

All mandatory assignments must be uploaded to Devilry.

- *The assignment must be submitted as a single PDF file.*
- *Scanned pages must be clearly legible. If these two requirements are not met, the assignment will not be assessed, but a new attempt may be given.*
- *The submission must contain your name, course and assignment number.*

Read the information about assignments carefully: http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html

To have a passing grade you must have satisfactory answers to at least 50% of the questions and have attempted to solve all of them.

PROBLEM 1

Let $\Omega \subset \mathbb{R}^n$ be connected, bounded and open. We call a function $u \in C^2(\overline{\Omega})$ *superharmonic* (in Ω) if $\Delta u \leq 0$ in Ω.

a)

Suppose u is superharmonic. Prove that

$$u(x) \geq \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy,$$

for any ball $B(x,r) \subset \Omega$.

b)

For a superharmonic function u prove that if

$$\min_{\overline{\Omega}} u = u(x_0),$$

for some $x_0 \in \Omega$ (interior minimum), then $u \equiv$ constant.

c)

Suppose $u, v \in C^2(\overline{\Omega})$ satisfy

$$\Delta u \leq 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega \quad \text{(superharmonic)}$$

and

$$\Delta v \geq 0 \text{ in } \Omega, \quad v = g \text{ on } \partial\Omega \quad \text{(subharmonic).}$$

Use part **b)** to show that

$$u(x) - v(x) \geq \min_{\partial\Omega}(f - g), \quad x \in \Omega.$$

d)

Let $\eta \in C^2(\mathbb{R})$ be a concave function. For a harmonic function u prove that $\eta(u)$ is a superharmonic function.

PROBLEM 2

a)

Consider the function

$$\Phi(x) = \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}}, \qquad x \neq 0,$$

where α_n denotes the volume of the unit ball in $\mathbb{R}^n$ ($n \geq 3$). Verify that

$$\Delta\Phi(x) = 0, \qquad x \neq 0.$$

Moreover, show that

$$|\Phi_{x_i x_i}(x)| \leq \frac{C}{|x|^n}, \qquad x \neq 0, \quad i = 1,\ldots,n\,.$$

b)

Show that the fundamental solution of the Laplace operator $-\Delta$ ($n \geq 3$) is

$$\Phi(x) = \frac{1}{n(n-2)\alpha_n}\frac{1}{|x|^{n-2}}, \qquad x \neq 0,$$

that is, show that

$$\int_{\mathbb{R}^n} \Phi(x)\Delta\phi(x)\,dx = -\phi(0), \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n).$$

c)

Let $u \in C^2(\mathbb{R}^n)$ solve the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Suppose u goes to zero as $|x| \to \infty$, sufficiently fast to justify integration by parts. Show that

$$f \in L^2(\mathbb{R}^n) \Longrightarrow \left\| u_{x_i x_j} \right\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}, \qquad i, j = 1,\ldots,n.$$

In other words, $f \in L^2(\mathbb{R}^n)$ implies $u \in H^2(\mathbb{R}^n)$ (regularization through the gain of two derivatives).

d)

Suppose $u \in \mathscr{A} := \left\{w \in C^2(\overline{\Omega}) : w = 0 \text{ on } \partial\Omega\right\}$ is a minimizer of the functional

$$I[w] := \int_\Omega \frac{1}{2}|Dw|^2 - F(w)\,dx, \qquad w \in \mathscr{A},$$

where $F : \mathbb{R} \to \mathbb{R}$ is a given C^1 function. Show that u satisfies the nonlinear Poisson equation (Euler-Lagrange equation)

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $f = F'$.

PROBLEM 3

a)

Define the weak derivative of a one-dimensional function $u : \mathbb{R} \to \mathbb{R}$. Compute the weak derivative of $u(x) = |x|$.

b)

Let Ω be a bounded open subset of $\mathbb{R}^n$ ($n > 2$). Define the Sobolev space $W^{1,2}(\Omega)$. What does it mean that $W^{1,2}(\Omega)$ is compactly embedded in $L^q(\Omega)$? For which values of q is the embedding compact?

c)

Consider the heat equation with a "friction" term:

$$u_t = \Delta u + cu \quad \text{in } \mathbb{R}^n \times (0,\infty),$$
$$u(x,0) = g(x), \quad x \in \mathbb{R}^n,$$

where c is a negative constant and the initial datum g is a continuous function. Use the energy method to prove that there exists at most one classical solution (decaying sufficiently fast towards zero as $|x| \to \infty$, to justify integration by parts).