UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MAT-INF3300/4300 — Partial differential equations and Sobolev spaces 1
Day of examination:	Wednesday, December 13, 2006.
Examination hours:	15.30 – 18.30.
This examination set consists of 2 pages.	
Appendices:	None
Permitted aids:	Approved calculator.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

Let $U = B(0,1) \setminus \{0\} \subset \mathbb{R}^2$, and consider the differential equation

$$\begin{cases} -\Delta u = 0 \quad x \in U \\ u|_{|x|=1} = 1, \quad u(0) = 0. \end{cases} \tag{1}$$

a) Show that if $u \in C^2(U) \cap C(\bar{U})$ is a solution of (1), then

$$v(r) = \frac{1}{2\pi r} \int_0^{2\pi} u(r,\theta)\, d\theta,$$

is another solution. Here $r = \sqrt{x_1^2 + x_2^2}$ and θ are polar coordinates in $\mathbb{R}^2$. Note that v is radially symmetric.

b) Show that there is *no* radially symmetric solution of (1). Why does this imply that (1) has no solution?

(Continued on page 2.)

Problem 2.

Let $U = B(0,1) \subset \mathbb{R}^2$. Show that $H_0^1(U) \not\subset C(U)$. (Hint: Consider the function $u(x) = \ln(|x|)$.)

Problem 3.

Let $U \subset \mathbb{R}^n$ be a bounded and simply connected domain such that ∂U is C^∞.

a) Let $f \in L^2(U)$, and let u be a weak solution of Laplace's equation

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2}$$

Let λ denote the *smallest* eigenvalue of $-\Delta$ in U, i.e., λ is the smallest number such that $-\Delta u = \lambda u$ has a nonzero solution $u \in H_0^1(U)$. We can write the solution u of (2) as $u = L(f)$. Show that

$$\sup_{f \in L^2(U)} \frac{\|L(f)\|_{L^2(U)}}{\|f\|_{L^2(U)}} \leq \frac{1}{\lambda}. \tag{3}$$

b) Now consider the *nonlinear* equation

$$\begin{cases} -\Delta u = \mu\,|u| + 1 & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{4}$$

where μ is a positive constant such that $\mu < \lambda$. In order to solve this we propose the following iterative scheme: $u^0 = 0$, and for $k > 0$ we let u^k be a weak solution of

$$\begin{cases} -\Delta u^k = \mu \left|u^{k-1}\right| + 1 & \text{in } U, \\ u^k = 0 & \text{on } \partial U. \end{cases}$$

Show that $\{u^k\}$ is a Cauchy sequence in $L^2(U)$, and thus convergent. Show that the limit is a weak solution of (4). You may find the following theorem useful.

Contraction mapping theorem. *Let B be a Banach space, and let $K : B \to B$ be a mapping such that for all x and y in B, we have*

$$\|K(x) - K(y)\| \leq \mu \|x - y\|, \textit{ where } \mu < 1.$$

Define $\{x_n\}_{n \geq 0}$ by $x_{n+1} = K(x_n)$, where x_0 is any element in B. Then $x_n \to \bar{x} \in B$, where $\bar{x}$ is the unique solution of $\bar{x} = K(\bar{x})$.

You do *not* have to prove this theorem.

c) If u is the weak solution of (4), show that $u \in C^\infty(U)$.

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