UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF4300 — Partial differential equations and Sobolev spaces I.

Day of examination: Monday, December 12, 2011.

Examination hours: 14.30 – 18.30.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let B be the unit ball of $\mathbb{R}^2$:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 < 1\} \tag{1}$$

a) Define the vectorfield $v : B \to \mathbb{R}^2$ by:

$$\forall x_1, x_2 \in \mathbb{R} \quad v(x_1, x_2) = (-x_2, x_1). \tag{2}$$

Find a solution u to the equation :

$$\left\{ \begin{aligned} -\Delta u + v \cdot \operatorname{grad} u &= 1, \quad \text{on } B. \\ u|_{\partial B} &= 0. \end{aligned} \right. \tag{3}$$

of the form:

$$u(x) = a|x|^2 + b, \tag{4}$$

for some $a, b \in \mathbb{R}$.

b) Define the vectorfield $w : B \to \mathbb{R}^2$ by:

$$\forall x_1, x_2 \in \mathbb{R} \quad w(x_1, x_2) = (x_2^2, x_1). \tag{5}$$

What is the divergence of w?

Fix $\delta > 0$. Write the weak formulation of the equation:

$$\left\{ \begin{aligned} -\delta\Delta u + w \cdot \operatorname{grad} u &= 1 \quad \text{on } B, \\ u|_{\partial B} &= 0. \end{aligned} \right. \tag{6}$$

Show that this formulation has a unique solution u in $\mathrm{H}_0^1(B)$.

(Continued on page 2.)

c) For any $\delta > 0$ we let $u_\delta \in \mathrm{H}^1_0(B)$ denote the solution to (6). Show that there exists $C > 0$ such that for all $\delta > 0$:

$$\|u_\delta\|_{\mathrm{H}^1(B)} \leq C/\delta. \tag{7}$$

Problem 2

Pick $p \in]1, +\infty[$ and define p' by $1/p' + 1/p = 1$.

a) Show that for any $u \in C^1_0(\mathbb{R})$ one can find $a \in \mathbb{R}$ such that for all $x \in \mathbb{R}$:

$$u(x)^p = \int_a^x p u(y)^{p-1} u'(y)\, \mathrm{d}\, y. \tag{8}$$

Deduce that there exists $C > 0$ such that for all $u \in C^1_0(\mathbb{R})$ we have:

$$\sup_{x \in \mathbb{R}} |u(x)| \leq C \|u\|^{1/p'}_{\mathrm{L}^p(\mathbb{R})} \|u'\|^{1/p}_{\mathrm{L}^p(\mathbb{R})}. \tag{9}$$

b) Pick $\theta \in [0, 1]$. We suppose that there exists $C > 0$ such that for all $u \in C^1_0(\mathbb{R})$:

$$\sup_{x \in \mathbb{R}} |u(x)| \leq C \|u\|^{1-\theta}_{\mathrm{L}^p(\mathbb{R})} \|u'\|^{\theta}_{\mathrm{L}^p(\mathbb{R})}. \tag{10}$$

Fix a non-zero $v \in C^1_0(\mathbb{R})$. For any $\delta \in \mathbb{R}$ with $\delta > 0$ define $v_\delta : \mathbb{R} \to \mathbb{R}$ by:

$$\forall x \in \mathbb{R} \quad v_\delta(x) = v(\delta x). \tag{11}$$

Show that:

$$\|v_\delta\|^{1-\theta}_{\mathrm{L}^p(\mathbb{R})} \|v'_\delta\|^{\theta}_{\mathrm{L}^p(\mathbb{R})} = \delta^{\theta - 1/p} \|v\|^{1-\theta}_{\mathrm{L}^p(\mathbb{R})} \|v'\|^{\theta}_{\mathrm{L}^p(\mathbb{R})}. \tag{12}$$

Deduce that $\theta = 1/p$.

c) Choose $a < b$ in $\mathbb{R}$ and let $I =]a, b[$. Deduce from (a) that there exists $C > 0$ such that for all $u \in W^{1,p}_0(I)$:

$$\|u\|_{\mathrm{L}^\infty(I)} \leq C \|u\|^{1/p'}_{\mathrm{L}^p(I)} \|u'\|^{1/p}_{\mathrm{L}^p(I)}. \tag{13}$$

Combine this inequality with the compactness of the injection $W^{1,p}_0(I) \to \mathrm{L}^p(I)$ to prove the following statement: Any bounded sequence in $W^{1,p}_0(I)$ has a subsequence which converges in $\mathrm{L}^\infty(I)$.

d) Fix a non-zero $w \in C^1_0(\mathbb{R})$. For any $n \in \mathbb{N}$, define $w_n : \mathbb{R} \to \mathbb{R}$ by:

$$\forall x \in \mathbb{R} \quad w_n(x) = w(x - n). \tag{14}$$

Show that the sequence $(w_n)_{n \in \mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R})$, but has no subsequence which converges in $\mathrm{L}^\infty(\mathbb{R})$.

END