UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT-MEK4270/9270 — Numerical methods for partial differential equations

Day of examination: 8 December 2023

Examination hours: 15:00 – 19:00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Function approximation

Let $V_N = \text{span}\{\psi_n(x)\}_{n=0}^N$ be a function space over the domain $\Omega = [0, 1]$ and let $\{\psi_n(x)\}_{n=0}^N$ be a set of basis functions that are orthogonal in the $L^2(\Omega)$ space. The $L^2(\Omega)$ inner product is defined as

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \tag{1}$$

for two real functions $f(x)$ and $g(x)$.

1a

Let $u(x)$ be any real function defined on the domain Ω. Describe the Galerkin method for approximating u with the expansion $u_N \in V_N$.

1b

Let $\psi_n(x) = P_n(\underline{x})$, where the reference coordinate $\underline{x} = 2x - 1 \in [-1, 1]$ and $P_n(\underline{x})$ is the n'th Legendre polynomial. The approximation $u_N \in V_N$ now implies that

$$u_N(x) = \sum_{i=0}^N \hat{u}_i P_i(\underline{x}(x)). \tag{2}$$

Find the expansion coefficients $\{\hat{u}_i\}_{i=0}^N$ expressed as inner products over $[-1, 1]$. The squared L^2 norm of the Legendre polynomials is $|P_n|^2 = (P_n, P_n)_{L^2([-1,1])} = \frac{2}{2n+1}$ for $n \geq 0$.

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1c

With the chosen Legendre basis, find the approximation to $u(x) = x(1-x)$ in V_N. Give the result as the expansion coefficients $\{\hat{u}_i\}_{i=0}^N$ and determine the smallest possible N for an exact approximation. The Legendre polynomials are defined as $P_0 = 1, P_1 = x$ and recursively

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad x \in [-1, 1], \tag{3}$$

for integer $n > 0$.

Problem 2 Ordinary differential equation

An ordinary differential equation is given as

$$u''(x) + \alpha u'(x) + \beta u(x) = f(x), \quad x \in \Omega = [-1, 1], \tag{4}$$

where $u(x)$ is the solution, $f(x)$ a real function and α and β are real constants. Assume at first homogeneous Dirichlet boundary conditions $u(\pm 1) = 0$ and that $V_N = \text{span}\{\psi_j(x)\}_{j=0}^N$ is an appropriate function space for approximations to the solution $u(x)$.

2a

Describe a Galerkin method that solves Eq. (4) for an approximation $u_N \in V_N$. Use integration by parts on the inner product containing the highest derivative.

2b

Describe the linear algebra problem that solves the variational problem defined in 2a. Suggest two different specific global bases $\{\psi_j(x)\}_{j=0}^N$, that both can be used to solve the described problem.

The finite element method

We will now make use of the finite element method in order to solve Eq. (4) with $\alpha = 0$. To this end the basis functions are chosen as piecewise linear

$$\psi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & x \in [x_{j-1}, x_j], \\ \frac{x - x_{j+1}}{x_j - x_{j+1}} & x \in [x_j, x_{j+1}], \\ 0, & x < x_{j-1} \text{ or } x > x_{j+1}, \end{cases} \tag{5}$$

where the mesh is $x_j = -1 + jh$ for $j = 0, 1, \ldots, N$, the element size $h = 2/N$, and $N + 1$ is the chosen number of mesh points. The element e is found in the interval $\Omega^{(e)} = [x_e, x_{e+1}]$ for $e = 0, 1, \ldots, N - 1$.

The mass matrix $A = (a_{ij})_{i,j=0}^N$ and stiffness matrix $S = (s_{ij})_{i,j=0}^N$ have components

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$$a_{ij} = \int_\Omega \psi_j(x)\psi_i(x)dx \quad \text{and} \quad s_{ij} = \int_\Omega \psi'_j(x)\psi'_i(x)dx, \tag{6}$$

respectively. These matrices are assembled as

$$A = \sum_{e=0}^{N-1} A^{(e)} \quad \text{and} \quad S = \sum_{e=0}^{N-1} S^{(e)}, \tag{7}$$

where $A^{(e)} = (a_{ij}^{(e)})_{i,j=0}^N$, $S^{(e)} = (s_{ij}^{(e)})_{i,j=0}^N$ and

$$a_{ij}^{(e)} = \int_{\Omega^{(e)}} \psi_j(x)\psi_i(x)dx \quad \text{and} \quad s_{ij}^{(e)} = \int_{\Omega^{(e)}} \psi'_j(x)\psi'_i(x)dx. \tag{8}$$

The element matrices $A^{(e)}$ and $S^{(e)}$ contain only 4 nonzero items each and we compute these nonzero items as

$$\tilde{a}_{rs}^{(e)} = \int_{\Omega^{(e)}} \psi_{q(e,s)}(x)\psi_{q(e,r)}(x)dx \quad \text{and} \quad \tilde{s}_{rs}^{(e)} = \int_{\Omega^{(e)}} \psi'_{q(e,s)}(x)\psi'_{q(e,r)}(x)dx, \tag{9}$$

where $(r,s) \in \{0,1\} \times \{0,1\}$ and $q(e,r) = e + r$ is a map from local index r on element e to global index $q(e,r)$.

The basis functions on each element are given as $\psi_{q(e,r)}(x) = \ell_r(\underline{x})$, where the Lagrange polynomials

$$\ell_0(\underline{x}) = \frac{1}{2}(1 - \underline{x}) \quad \text{and} \quad \ell_1(\underline{x}) = \frac{1}{2}(1 + \underline{x}), \tag{10}$$

for the reference coordinate $\underline{x} \in [-1, 1]$. The linear map from $\underline{x}$ to x on any element e can be written as

$$x = \frac{1}{2}(x_e + x_{e+1}) + \frac{h}{2}\underline{x}. \tag{11}$$

2c

Write the local element matrices $\tilde{a}_{rs}^{(e)}$ and $\tilde{s}_{rs}^{(e)}$ as integrals over the reference domain $[-1, 1]$ and compute all elements of these matrices. (Hint: The matrices are symmetric.)

2d

Assume Neumann boundary conditions $u'(-1) = a$ and $u'(1) = b$. Describe how Eq. (4) (with $\alpha = 0$) can be solved with the FEM using the Lagrange polynomials (10) for the basis functions. Describe also how the finite element solution $u_N(x) \in V_N$ can be evaluated for any single point $x \in [-1, 1]$.

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Problem 3 Time-dependent wave equation

Consider the time-dependent wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x, t \in [0, L] \times [0, T], \tag{12}$$

where c, T and L are positive constants. Equation (12) is solved with two suitable boundary conditions at $x = 0$ and $x = L$ and initial conditions $u(x, 0) = I(x)$ for some real function $I(x)$ and $\frac{\partial u(x,0)}{\partial t} = 0$.

A central finite difference method for solving the time-dependent wave equation can be written for all internal mesh points as

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \tag{13}$$

where time is discretized as $t_n = n\Delta t$, space as $x_j = j\Delta x$ and Δt and Δx are assumed to be constant. In vector form we write $u^n = \{u_j^n\}_{j=0}^N$ and thus Eq. (13) can be written as

$$u^{n+1} - 2u^n + u^{n-1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 D^{(2)} u^n, \tag{14}$$

where the matrix $D^{(2)} = (d_{ij}^{(2)})_{i,j=0}^N$ is

$$d_{ij}^{(2)} = \begin{cases} 1, & j = i \pm 1, \\ -2, & j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

for $i = 1, 2, \ldots, N-1$, and with appropriate modifications to rows $i = 0$ and $i = N$ to accommodate boundary conditions.

3a

Describe $\Delta t, \Delta x, u_j^n$, and how the two initial conditions can be specified.

3b

Assume Dirichlet boundary conditions $u(0, t) = u(L, t) = 0$ and describe the complete solution algorithm for the wave equation.

3c

Assume Neumann boundary conditions $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$ and describe the complete solution algorithm for the wave equation.