Trial exam 2023 MATMEK-4270

1) a) Find $u_N \in V_N$ such that

(1) $$(u-u_N, v)_{L^2} = 0 \,, \quad \forall v \in V_N$$

The L^2 inner product is here defined as

$$(a,b)_{L^2([-1,1])} = \int_{-1}^{1} a\, b\, dx \,,$$

where $a(x)$ and $b(x)$ are real functions.

1b) Insert for $v = P_i$ and $u_N = \sum_{j=0}^{N} \hat{u}_j P_j$ in (1)

$$\left(u - \sum_{j=0}^{N} \hat{u}_j P_j, P_i\right) = 0 \,, \quad \forall i = 0,1,\dots,N$$

$$\sum_{j=0}^{N} (P_j, P_i)\hat{u}_j = (u, P_i)$$

1b) Use $(P_j, P_i) = \frac{2}{2i+1}\delta_{ij}$ and solve for $\hat{u}$

$$\hat{u}_i = \frac{2i+1}{2}(u, P_i) \quad , \quad i = 0, 1, \ldots, N$$

Insert for $u = x^3$

$$② \qquad \hat{u}_i = \frac{2i+1}{2}(x^3, P_i) \, , \quad i = 0, 1, \ldots, N$$

Since x^3 is an odd function all even coefficients $\hat{u}_{2i} = 0$, $i = 0, 1, \ldots$

Since $x^3 \in \mathbb{P}_3$ $N = 3$. So need to compute $\hat{u}_1$ and $\hat{u}_3$ from ②.

$$\hat{u}_1 = \frac{2\cdot 1+1}{2}\int_{-1}^{1} x^3 \cdot x \, dx = \frac{3}{2}\left[\frac{1}{5}x^5\right]_{-1}^{1}$$

$$\hat{u}_1 = \underline{\underline{\frac{3}{5}}}$$

For $\hat{u}_3$ we need P_3, which can be computed from the recursion in Eq. 1.

7b) $P_3 = \frac{1}{2}(5x^3-3x)$

$$\hat{u}_3 = \frac{2\cdot 3+1}{2}\int_{-1}^{1} x^3\cdot\frac{1}{2}(5x^3-3x)\,dx$$

$$= \frac{7}{4}\int_{-1}^{1} 5x^6-3x^4\,dx$$

$$= \frac{7}{4}\left[\frac{5}{7}x^7-\frac{3}{5}x^5\right]_{-1}^{1} = \frac{7}{4}\left(\frac{5}{7}-\frac{3}{5}\right)\cdot 2$$

$$= \frac{1}{2}\left(5-\frac{21}{5}\right) = \underline{\underline{\frac{2}{5}}}$$

$$\underline{\underline{\hat{u} = \left\{0, \frac{3}{5}, 0, \frac{2}{5}\right\}}}$$

Note that

$$u_N = \sum_{j=0}^{3}\hat{u}_j P_j = x^3$$

but the expansion coefficients are here required for full score, since these are requested.

1c) The approximation is exact

$$u_N = x^3$$

$\{x^n\}_{n=0}^{N}$ is a poor basis since the basis functions are not orthogonal in the L^2 inner product space, or any other space. The basis functions are almost linearly dependent and the mass matrix

$$M = (m_{ij})_{i,j=0}^{N}, \quad m_{ij} = (x^j, x^i)$$

is ill-conditioned, leading to large round-off errors for numerical computations with finite precision.

2a) The Galerkin method is to find $u_N \in V_N$, such that

$$\text{ⓐ} \quad (u_N'', v) + \alpha^2 (u_N, v) = (f, v)$$

for all $v \in V_N$. Alternatively, replace ⓐ by ⓑ:

$$\text{ⓑ} \quad -(u_N', v') + \alpha^2 (u_N, v) = (f, v) - [u_N' v]_{x=-1}^{x=1}$$

where $[u_N' v]_{-1}^{1} = 0$, since $v(\pm 1) = 0$.

2b) Insert for $v = \varphi_i$ and $u_N = \sum_{j=0}^{N} \hat{u}_j \varphi_j$ in either ⓐ or ⓑ. Using ⓐ:

$$\sum_{j=0}^{N} (\varphi_j'', \varphi_i) \hat{u}_j + \sum_{j=0}^{N} \alpha^2 (\varphi_j, \varphi_i) \hat{u}_j = (f, \varphi_i)$$

for $i = 0, 1, \dots, N$.

Specific bases that can be used:

$$\{ P_i - P_{i+2} \}_{i=0}^{N}$$

$$\left\{ \sin\left((i+1)\pi \frac{(x+1)}{2} \right) \right\}_{i=0}^{N}$$

Both satisfy the boundary conditions.

2b) Note that this problem also can be solved with the finite element method. However, since it is given that $\varphi_j(\pm 1) = 0$, this means that you cannot use Lagrange polynomials, since $l_0(-1) = 1$ and $l_N(1) = 1$.

2c) Define

$$u = \hat{u} + B,$$

where $B(-1) = u(-1) = a$ and $B(1) = u(1) = b$:

$$B = \frac{a}{2}(1-x) + \frac{b}{2}(1+x)$$

Insert $u = \hat{u} + B$ into Eq. (2):

$$(\hat{u} + B)'' + \alpha^2(\hat{u} + B) = f$$

$$\hat{u}'' + \alpha^2 \hat{u} = f - \alpha^2 B$$

2c) The Galerkin problem becomes:
Find $\tilde{u}_N \in V_N$ such that

$$(\tilde{u}_N'', v) + \alpha^2(\tilde{u}_N, v) = (f - \alpha^2 B, v)$$

for all $v \in V_N$.

Set $u_N = \tilde{u}_N + B$

2d) The basis functions do <u>not</u> satisfy $\varphi_j(\pm 1) = 0$, or $\varphi_j'(\pm 1) = 0$.
Hence we need to set the boundary conditions weakly using integration by parts.
From 2a) (B):

$$-(u_N', v') + \alpha^2(u_N, v) = (f, v) - [u_N' v]_{x=-1}^{x=1}$$

Since $u_N'(\pm 1) = 0$, we can solve

$$-(u_N', v') + \alpha^2(u_N, v) = (f, v)$$

2d) Insert for $v = P_i$ and $u_N = \sum_{j=0}^{N} \hat{u}_j P_j$ to obtain the equation

$$-\sum_{j=0}^{N} (P_j', P_i')\hat{u}_j + \alpha^2 \sum_{j=0}^{N} (P_j, P_i)\hat{u}_j = (f, P_i)$$

for $i = 0, 1, \ldots, N$.

Note that it is not required to derive the linear algebra problem.

3a) The diffusion equation is discretized using N equal intervals of size $\Delta x = \frac{1}{N}$, and N_t time steps of length $\Delta t = \frac{T}{N_t}$.

A mesh function is defined as

$$u_j^n = u(x_j, t_n) = u(j\Delta x, n\Delta t)$$

Initialize $u^0 = \{u_j^0\}_{j=0}^{N}$ as

$$u_j^0 = I(x_j) \quad , \; j = 0, 1, \ldots, N$$

Discretization:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = C \; \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} + u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{2\Delta x^2}$$

Or in matrix form:

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{C}{2} D^{(2)}(u^{n+1} + u^n) \qquad *$$

3a) where $D^{(2)}$ is the central, second order accurate, second differentiation matrix. $D^{(2)} \in \mathbb{R}^{N \times N}$

We get from *

$$\left(I - \frac{c\Delta t}{2} D^{(2)}\right) u^{n+1} = \left(I + \frac{c\Delta t}{2} D^{(2)}\right) u^{n}$$

where $I \in \mathbb{R}^{N \times N}$ is the identity matrix.

Writing furter

$$A u^{n+1} = B u^{n}$$

where $A = I - \frac{c\Delta t}{2} D^{(2)}$, $B = I + \frac{c\Delta t}{2} D^{(2)}$,

we get

$$u^{n+1} = A^{-1} B u^{n} \quad , \quad n = 0, 1, \dots, N_t - 1.$$

The complete algorithm becomes:

Initialize u^0

For $n = 0, 1, \dots, N_t - 1$, find

$u^{n+1} = A^{-1} B u^{n}$

Set $u_0^{n+1} = 0$ and $u_N^{n+1} = 0$

3b) Define the approximation

$$u_N^n = \sum_{j=0}^{N} \hat{u}_j^n \psi_j(x) \approx u(x, n\Delta t)$$

and the function space $V_N = \text{span}\{\psi_j\}_{j=0}^{N}$.

Let $\psi_j(0) = \psi_j(1) = 0$ for all $j \geq 0$.

Initialize u_N^0 by projection: Find $u_N^0 \in V_N$, such that

$$(I(x) - u_N^0, v) = 0, \quad \forall v \in V_N$$

For remaining time steps:

For $n = 0, 1, \ldots, N_t - 1$ find $u_N^{n+1} \in V_N$ such that

$$(u_N^{n+1} - u_N^n, v) = \frac{c\Delta t}{2}\left(\frac{\partial^2}{\partial x^2}(u_N^{n+1} + u_N^n), v\right)$$

for all $v \in V_N$.

3c) We have the equation

$$\left(I - \frac{c\Delta t}{2} D^{(2)}\right) u^{n+1} = \left(I + \frac{c\Delta t}{2} D^{(2)}\right) u^{n} \quad \otimes$$

Assume that the solution can be written as (ansatz)

$$u^{n+1} = g^{n+1} u^{0},$$

where g is the amplification factor, independent of x and t.
Insert the ansatz into $\otimes$ and divide by g^n:

$$g\left(I - \frac{c\Delta t}{2} D^{(2)}\right) u^{0} = \left(I + \frac{c\Delta t}{2} D^{(2)}\right) u^{0}$$

We can now use that

$$D^{(2)} u^{0} = \lambda u^{0},$$

where λ are the eigenvalues of $D^{(2)}$.
We get

$$g\left(I - \frac{c\Delta t}{2} \lambda I\right) u^{0} = \left(I + \frac{c\Delta t}{2} \lambda I\right) u^{0}$$

3c) Simplify by eliminating u^0

$$g\left(1-\frac{c\Delta t}{2}\lambda\right)=\left(1+\frac{c\Delta t}{2}\lambda\right),$$

such that

$$g=\frac{1+\frac{c\Delta t}{2}\lambda}{1-\frac{c\Delta t}{2}\lambda}$$

It is reported that $\lambda<0$ and real, and also that $c>0$. Hence $|g|\leq 1$ for all Δt, and thus unconditionally stable.