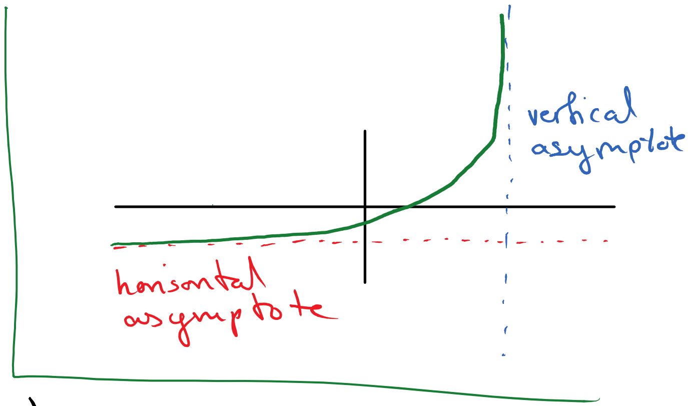
Oppg. 42d (asymptoter)

$f(x) = \frac{1}{x(x+2)}$ drøft, skisser og finn asymptoter




## Vertikale asymptoter:

(vi ser etter disse der f ikke er definert, eller ikke er kontinuerlig)

- f ikke definert for $x = 0$ og $x = -2$

$$\lim_{x \to 0^+} \frac{1}{x} \cdot \frac{1}{x+2} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} \cdot \frac{1}{x+2} = -\infty$$

$x = 0$ er vertikal asymptote

$$\lim_{x \to -2^+} \frac{1}{x} \cdot \frac{1}{x+2} = -\infty$$

$$\lim_{\to -2^-} \frac{1}{x} \cdot \frac{1}{x+2} =$$

$x = 2$ er vertikal asymptote

## Horisontale asymptoter

$$\lim_{x \to \infty} \frac{1}{x} \cdot \frac{1}{x+2} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} \cdot \frac{1}{x+2} = 0$$

$y = 0$ er en horisontal asymptote

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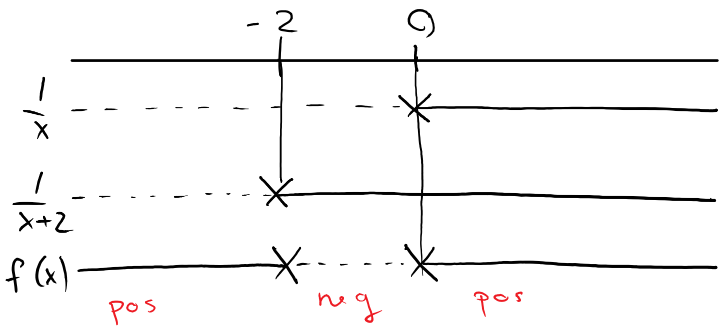
$f(x) = \frac{1}{x} \cdot \frac{1}{x+2}$



# Oppg. 42d (fortsetter)

$$f(x)=\frac{1}{x(x+2)}=\underbrace{x^{-1}}_{u}\cdot\underbrace{(x+2)^{-1}}_{v}$$

$u'=-1x^{-2}$

$v'=-1(x+2)^{-2}$

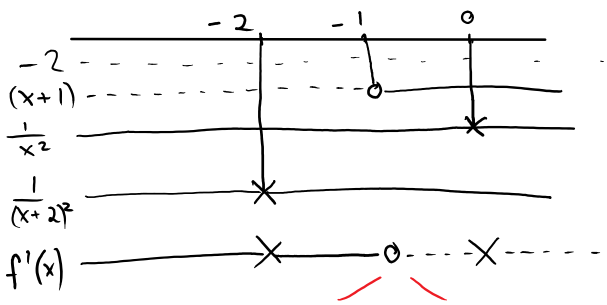
Vi kan derivere vha. produkt-regelen $(uv)'=u'v+uv'$ !

$$f'(x)=u'v+uv'=-x^{-2}(x+2)^{-1}+x^{-1}(-1)(x+2)^{-2}$$

$$=\frac{-1}{x^2(x+2)}+\frac{-1}{x(x+2)^2}$$

$$=\frac{-(x+2)}{x^2(x+2)}+\frac{-x}{x(x+2)^2}$$

$$=\frac{-2(x+1)}{x^2(x+2)^2}$$




vi har lokalt max-punkt i $x=-1$
og $f(-1)=-1$

Finne $f''(x)$.

metode 1: Bruk kvotientregelen på $\frac{-2(x+1)}{x^2(x+2)^2}$

metode 2: Deriver $=-x^{-2}(x+2)^{-1}+x^{-1}(-1)(x+2)^{-2}$
vha. produktregelen.

- Det viser seg å være minst jobb med metode 2:

$$f''(x)=-\left(\underbrace{x^{-2}}_{u}\underbrace{(x+2)^{-1}}_{v}\right)'-\left(\underbrace{x^{-1}}_{g}\underbrace{(x+2)^{-2}}_{h}\right)'$$

$u'=-2x^{-3}$

$v'=-(x+2)^{-2}$

$g'=-x^{-2}$

$h'=-2(x+2)^{-3}$

$$=-(u'v+uv')-(g'h+gh')$$

$$=-\left(-2x^{-3}(x+2)^{-1}-x^{-2}(x+2)^{-2}\right)$$

$$-\left(-x^{-2}(x+2)^{-2}-x^{-1}\cdot 2(x+2)^{-3}\right)$$

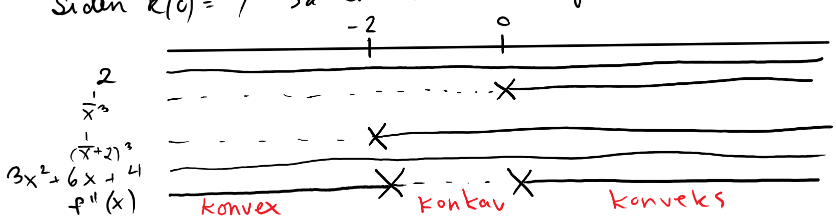
$$=\frac{2}{x^3(x+2)}+\frac{1}{x^2(x+2)^2}+\frac{1}{x^2(x+2)^2}+\frac{2}{x(x+2)^3}$$

$$=\frac{2(x+2)^2}{x^3(x+2)^3}+\frac{2x(x+2)}{x^3(x+2)^3}+\frac{2x^2}{x^3(x+2)^3}$$

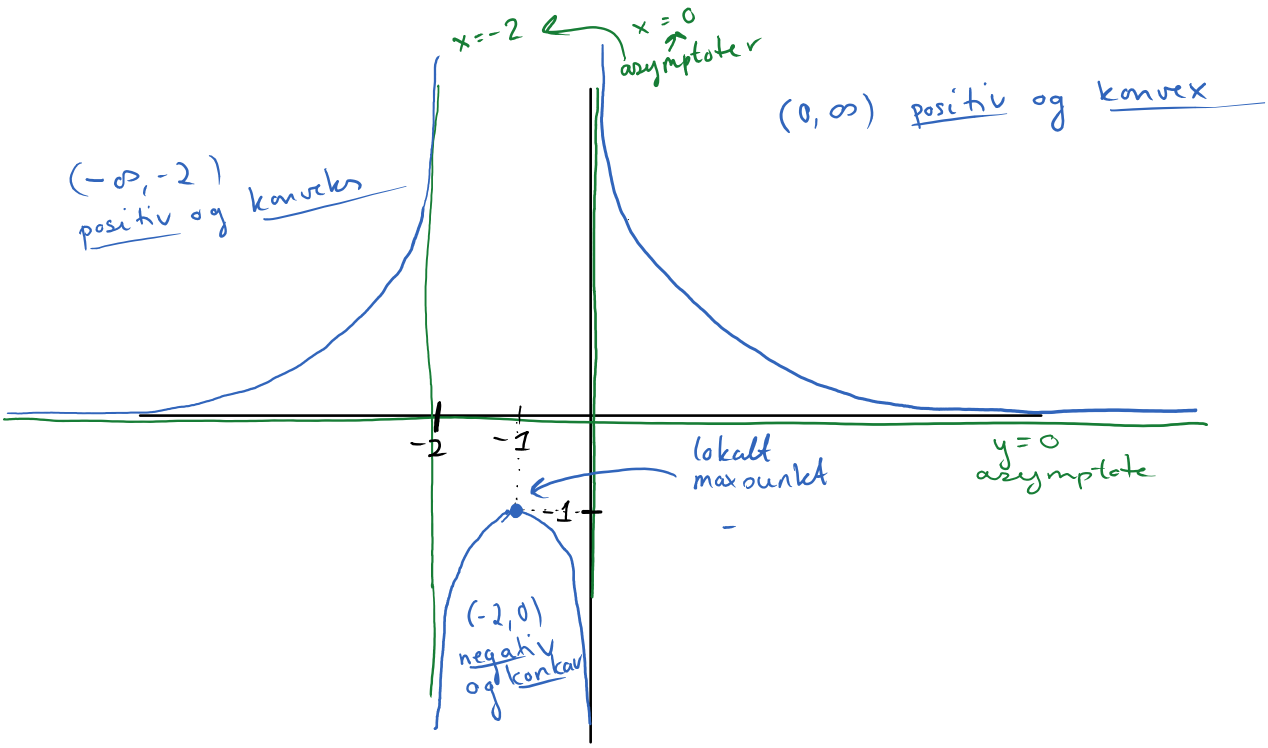
$$=\frac{(2x^2+8x+8)+(2x^2+4x)+2x^2}{x^3(x+2)^3}$$

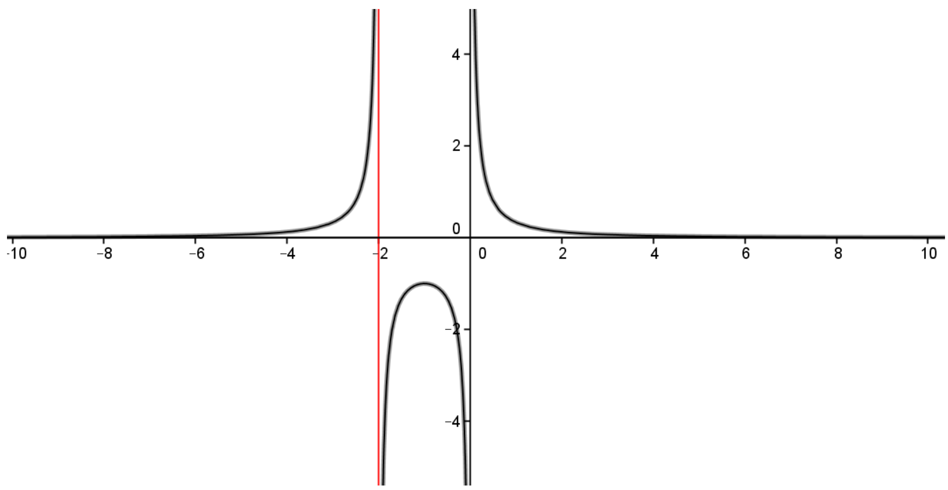
$$=\frac{6x^2+12x+8}{x^3(x+2)^3}=\frac{2}{x^3(x+2)^3}(3x^2+6x+4)$$

Merk at $k(x)=3x^2+6x+4$ ikke har nullpunkter siden
$b^2-4ac=36-48<0$
Siden $k(0)=4$ så er $k$ alltid positiv (Hvorfor ?)



Oppg. 42d (fortsetter)

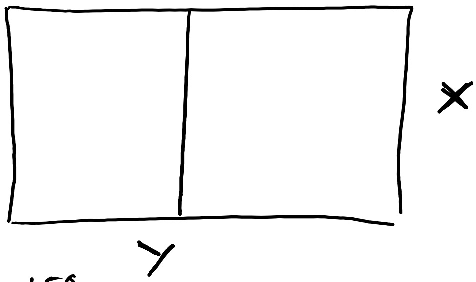


# Oppg. 46

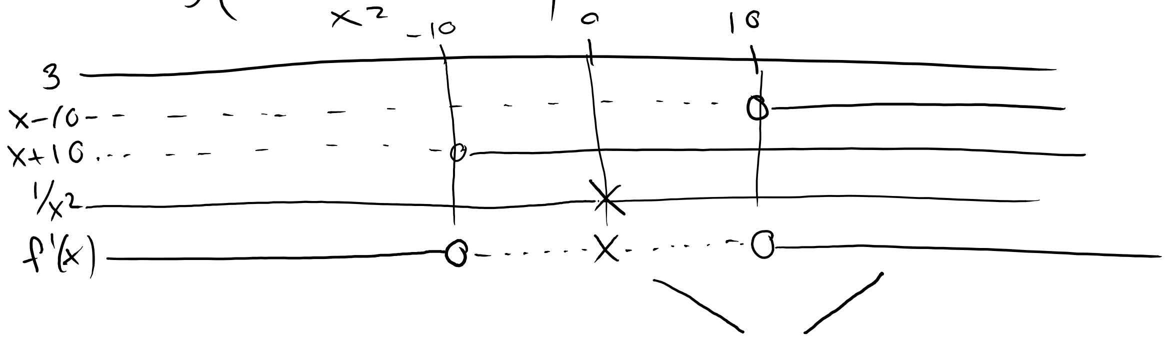
Vi vil gjerde inn et rektangulært område som skal ha areal 150 m^2. I tillegg vil dele området i to med et gierde som går parallelt med en avsidene til området. Vi ønsker å bruke minst mulig materialer, dvs. vi ønsker å et kortest mulig gjerde. Hva må lengdene på sidene til området være for å få til dette?

$$x \cdot y = 150$$
$$y = \frac{150}{x}$$




$$f(x) = 2y + 3x = 2 \cdot \frac{150}{x} + 3x$$
$$= \frac{300}{x} + 3x$$
$$f'(x) = 300\left(\frac{-1}{x^2}\right) + 3 = 3\left(1 - \frac{100}{x^2}\right)$$
$$= 3\left(\frac{x^2}{x^2} - \frac{100}{x^2}\right) = 3\left(\frac{x^2 - 100}{x^2}\right)$$
$$= 3\left(\frac{(x-10)(x+10)}{x^2}\right)$$

$$\boxed{\begin{aligned} &a^2 - b^2 \\ &= (a-b)(a+b) \end{aligned}}$$




$x = 10$ og $y = \frac{150}{10} = 15$ gir kortest mulig gjerde.

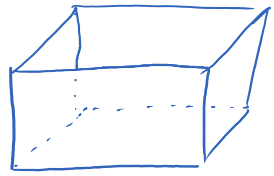
Oppg. 48

papp-boks uten lokk (rette vinkler)

Volum 108 $cm^3$

kvadratisk bunn

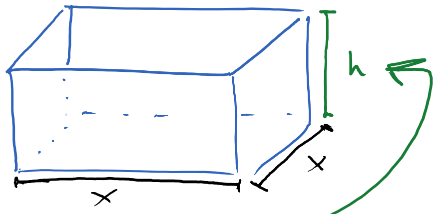
Vil bruke minst mulig papp (målt i areal)



A(hele boksen) = A(bunn) + A(sidene)

$V = x \cdot x \cdot h = x^2 h = 108$

$h = \frac{108}{x^2}$




$A(x) = x \cdot x + 4(x \cdot h)$

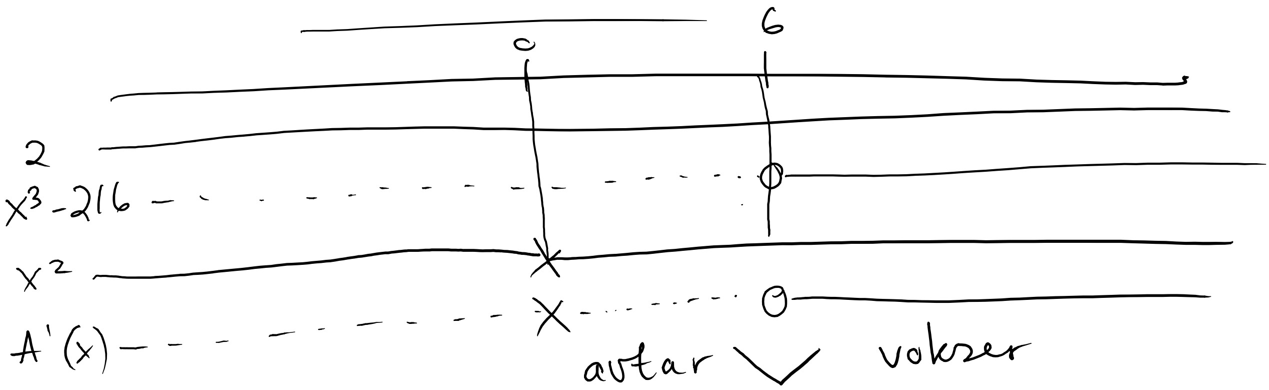
$= x^2 + 4\left(x \cdot \frac{108}{x^2}\right) = x^2 + \frac{432}{x}$

$A'(x) = 2x + 432 \cdot \left(-\frac{1}{x^2}\right) = \frac{2x^3 - 432}{x^2} = \frac{2(x^3 - 216)}{x^2}$

Setter $A'(x) = 0$ for å finne mulig nullpunkt

$x^3 - 216 = 0$

$x = \sqrt[3]{216} = 6$




$x = 6$ er min-punkt

Vi bruker minst papp ved å velge

$x = 6$ og $h = \frac{108}{x^2} = \frac{108}{36} = 3$

Oppg. 49c

$$1+4+9+16+\dots+256 = 1^2+2^2+3^2+4^2+\dots+16^2 = \underline{\underline{\sum_{n=1}^{16} n^2}}$$

Oppg. 51

La $S_n = a + ar + ar^2 + \dots + ar^n$ (endelig geometrisk rekke)

a) skriv ut leddene i rekken $r \cdot S_n$

$$r \cdot S_n = r \cdot (a + ar + ar^2 + \dots + ar^n)$$
$$= \underline{\underline{ar + ar^2 + ar^3 + \dots + ar^{n+1}}}$$

b) finn $rS_n - S_n$

$$rS_n - S_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + ar^{n+1} - (a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^n})$$
$$= ar^{n+1} - a = \underline{\underline{a(r^{n+1} - 1)}}$$

c) Vis at $S_n = a \dfrac{1 - r^{n+1}}{1 - r}$ hvis $r \neq 1$

Hva er $S_n$ hvis $r = 1$ ?

$\underline{r \neq 1}$:

$$r S_n - S_n = a(r^{n+1} - 1)$$
$$S_n (r - 1) = a(r^{n+1} - 1)$$
$$S_n = \frac{a(r^{n+1} - 1)}{r - 1}$$
$$= a \frac{(1 - r^{n+1})}{1 - r}$$

$\underline{r = 1}$: $S_n = a + a \cdot 1 + a \cdot 1^2 + \dots + a 1^n$
$$= \underline{\underline{a(n+1)}}$$

d) Finn $1 + 2 + 4 + 8 + \dots + 128$

$$= \underset{a}{1} + 1 \cdot \underset{r}{2} + 1 \cdot 2^2 + 1 \cdot 2^3 + \dots + 1 \cdot 2^{\overset{n}{7}}$$
$$= 1 \cdot \frac{(1 - 2^8)}{1 - 2} = \frac{1 - 256}{-1} = \underline{\underline{255}}$$

e) $S = a + ar + ar^2 + ar^3 + \dots$ (uendelig geometrisk rekke altså $S = \lim_{n \to \infty} S_n$)

Hvis $r < 1$, så er $S = \dfrac{a}{1 - r}$

Finn $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

$$= \underset{a}{\frac{1}{2}} + \frac{1}{2} \underset{r}{\left(\frac{1}{2}\right)} + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} \left(\frac{1}{2}\right)^4 + \dots$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \underline{\underline{1}}$$

Oppg. 56

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ Finn følgende bestemte integral

$F(x)$ $f(x)$ $\int_3^4 \frac{1}{x^2}\,dx = -\int_3^4 \left(-\frac{1}{x^2}\right)dx = -\int_3^4 f(x)\,dx$

$= -\left(F(4) - F(3)\right)$

setning 18

$= -\left(\frac{1}{4} - \frac{1}{3}\right) = \underline{\underline{\frac{1}{12}}}$

Oppg. 62c (substitusjon)

Finn $\int \frac{x^2}{\sqrt{x^3+1}}\,dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{3}\,du$

$$\boxed{\begin{aligned} u &= x^3+1 \\ \frac{du}{dx} &= 3x^2 \\ \frac{1}{3}\,du &= x^2\,dx \end{aligned}}$$

$$= \frac{1}{3}\int u^{-1/2}\,du$$

$$= \frac{1}{3}\left(2u^{\frac{1}{2}} + C\right)$$

$$= \frac{2}{3}u^{1/2} + \frac{C}{3}$$

$$= \underline{\underline{\frac{2}{3}\sqrt{x^3+1} + C'}}$$

Oppg. 64b (delvis integrasjon)

$$\boxed{\int u v' \, dx = uv - \int u' v \, dx}$$

Finn $\displaystyle\int \frac{x}{(x+2)^{1/3}} \, dx = \frac{3x}{2}(x+2)^{2/3} - \int \frac{3}{2}(x+2)^{2/3} \, dx$

$$\boxed{\begin{array}{ll} u = x & u' = 1 \\ v' = (x+2)^{-1/3} & v = \frac{3}{2}(x+2)^{2/3} \end{array}}$$

$$= \frac{3x}{2}(x+2)^{2/3} - \frac{3}{5} \cdot \frac{3}{2}(x+2)^{5/3} + C$$

$$= \frac{3x}{2}(x+2)^{2/3} - \frac{9}{10}(x+2)^{5/3} + C$$

Oppg. 65

a) Finn integralet $\int x^3\sqrt{x^2+1}\,dx$ ved å bruke delvis integrasjon ($u = x^2$, $v' = x\sqrt{x^2+1}$) og substitusjon.

b) Finn det bestemte integralet $\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx$

$$\int x^2 \cdot x\sqrt{x^2+1}\,dx$$

$$\boxed{\begin{array}{l} u = x^2, \quad u' = 2x \\ v' = x(x^2+1)^{1/2}, \quad v = \frac{1}{3}(x^2+1)^{3/2} \quad \text{sjekk! (kjerneregelen)} \end{array}}$$

$$= x^2 \cdot \frac{1}{3}(x^2+1)^{3/2} - \int \frac{2x}{3}(x^2+1)^{3/2}\,dx$$

$$\boxed{\begin{array}{l} u = x^2+1 \\ \frac{du}{dx} = 2x \\ du = 2x \end{array}}$$

$$= \frac{x^2}{3}(x^2+1)^{3/2} - \int \frac{u^{3/2}}{3}\,du$$

$$= \text{---} \,\text{''}\, \text{---} - \frac{2}{5}\cdot\frac{1}{3}u^{5/2} + C$$

$$= \frac{x^2}{3}(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + C$$

b) $$\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx = \left[\frac{x^2}{3}(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2}\right]_0^{\sqrt{3}}$$

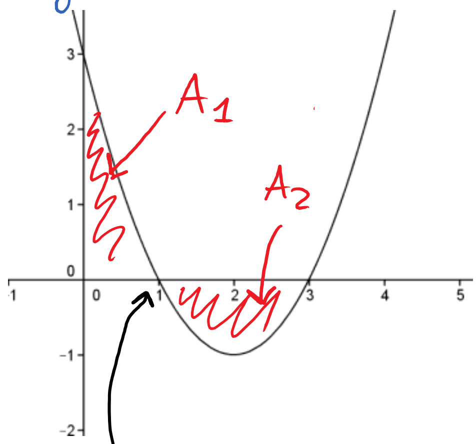
$$= \left(\frac{\cancel{3}}{\cancel{3}}(3+1)^{\frac{3}{2}} - \frac{2}{15}(3+1)^{\frac{5}{2}}\right) - \left(0 - \frac{2}{15}1^{5/2}\right)$$

$$= 2^3 - \frac{2}{15}\cdot 2^5 + \frac{2}{15} = \underline{\underline{\frac{58}{15}}}$$

Oppg. 66b (areal)

$h(x) = x^2 - 4x + 3$ , $x \in [0,3]$

skisser området avgrenset av funksjonen og x-aksen på det gitte intervallet. Finn arealet.




$$A_1 = \int_0^1 h(x)\,dx$$

$$A_2 = -\int_1^3 h(x)\,dx$$

$$A = A_1 + A_2$$

$$\int h(x)\,dx = \frac{1}{3}x^3 - 2x^2 + 3x + C$$

$$A_1 = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_0^1$$
$$= \left(\frac{1}{3} - 2 + 3\right) - 0 = \underline{\frac{4}{3}}$$

$$A_2 = -\left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$$
$$= -\left(\left(\frac{1}{3}3^3 - 2\cdot 3^3 + 3\cdot 3\right) - \frac{4}{3}\right)$$
$$= -\left(0 - \frac{4}{3}\right) = \underline{\frac{4}{3}}$$

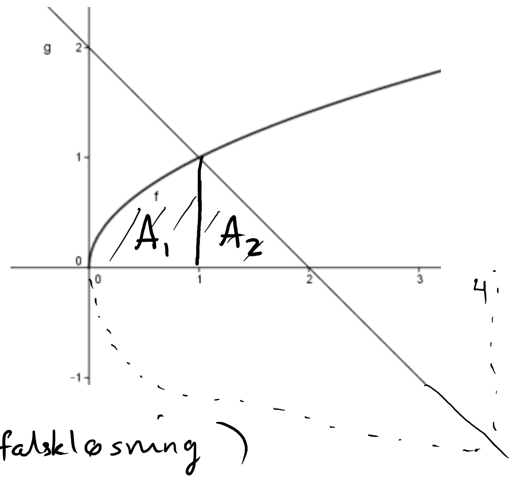
$$A = 2\cdot\frac{4}{3} = \underline{\underline{\frac{8}{3}}}$$

Vi finner skjæringspunktet mellom h og x aksen:

$$x = \frac{4 \pm \sqrt{4^2 - 4\cdot 3}}{2}$$
$$= \begin{cases} 3 \\ 1 \end{cases}$$

Oppg. 67a

Skisser og finn areal til området avgrenset av x-aksen, $f(x) = \sqrt{x}$ og $g(x) = -x+2$




Finner skjæringspunktet mellom f og g

$$\sqrt{x} = -x+2$$
$$x = (-x+2)^2$$
$$x = x^2 - 4x + 4$$
$$x^2 - 5x + 4 = 0$$
$$x = \frac{5 \pm \sqrt{5^2 - 4 \cdot 4}}{2} = \frac{5 \pm 3}{2} = \begin{cases} 4 \\ 1 \end{cases}$$ (falskløsning)

$$A_1 = \int_0^1 f(x)\,dx, \quad A_2 = \int_1^2 g(x)\,dx$$

$$A_1 = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

$$A_2 = \int_1^2 -x + 2\,dx = \left[-\frac{1}{2}x^2 + 2x\right]_1^2 = \left(-\frac{1}{2} \cdot 2^2 + 2 \cdot 2\right) - \left(-\frac{1}{2} \cdot 1^2 + 2 \cdot 1\right)$$

$$= \frac{1}{2}$$

$$A = A_1 + A_2 = \frac{2}{3} + \frac{1}{2} = \underline{\underline{\frac{7}{6}}}$$