MAT 1001, fall 2015
Compulsory Exercise 2 (Oblig 2)

Deadline for hand-in: Thursday, November 6th, 2:30 PM

You are allowed to work together on the exercises, but everyone has to handle inn their own solution. The assignment must be handed in in the special box at the floor marked "7" in Niels Henrik Abels hus (math building) before the deadline. Remember to fill in and attach a front page - front pages are found nearby the box, or on the Internet.

Each exercise a) - k) counts 0-10 points, all together a maximum of 110 points. Your Oblig 2 is approved if you score 20 points or more on Part 1 (50%), and 40 points or more on Part 2 (a bit more than 50%).

Introduction

According to Einstein's general theory of relativity, the path traversed by a ray of light (a photon) passing by a star (in our case, the Sun), is given by the differential equation:

$$\frac{\left(\frac{dr}{dt}\right)^2 - E^2}{1 - \frac{R_S}{r}} + \frac{L^2}{r^2} = 0$$

where $r = r(t)$ is the photon's distance to the center of the star (as a function of time, t), R_S is the so-called Schwarzschild radius[1], E is the energy of the photon and L is called the photon's angular momentum.

Throughout the exerscise the magnitudes R_S, E and L will be considered as constants. We shall accept the given differential equation without any further argueing about how it is deduced.

We will solve this equation and find out how the Sun's gravity affects the path of the light ray when it passes close to the Sun (the fact that light is also affected by gravity is an important result in the theory of relativity).

NOTE: You can solve the exercises in any order and simply use the results from exercises you haven't solved yet. This is part of the reason why the exercises focus on you showing how one derives the answer, and not the answer itself. This has the added benefit that you will always to be able to check that your answer is correct. But the main point is how you use mathematics to arrive at the desired result.

Good luck!

[1] If we compress the star so this becomes its new radius (3 km for a Sun-sized star), it will collapse and become a black hole!

Part 1

In the first part of the exercise, problems a)-d), the challenge is to use Leibniz notation for the derivative to manipulate magnitudes and variables to simplify the differential equation. Remember that the chain rule tells us that for a composed function $r(\phi(t))$, the derivative of the function r with respect to t is given by

$$\frac{dr}{dt} = \frac{dr}{d\phi} \cdot \frac{d\phi}{dt}$$

a) In the equation on the previous page, L is defined as follows:

$$L := r^2 \frac{d\phi}{dt}$$

The angle ϕ is explained in figures 1 and 2. It varies with the distance, and is a function of t. Use this (and the chain rule) to show that

$$\frac{dr}{dt} = \frac{dr}{d\phi} \cdot \frac{L}{r^2}$$

b) In the following, we will make a substitution to simplify the calculations:

$$u := \frac{1}{r}$$

i.e. the variable u expresses the inverse distance. Far away it is close to 0, and it reaches its maximum value when the photon reaches its minimum distance to the Sun. Thus the magnitude u can be considered as a function of the distance r, and therefore also as a function of the angle ϕ, through $r = r(\phi)$.

Apply this substitution to the result from a), and show that

$$\frac{dr}{dt} = -L \frac{du}{d\phi}$$

c) Insert the result from b) into the differential equation

$$\frac{\left(\frac{dr}{dt}\right)^2 - E^2}{1 - \frac{R_S}{r}} + \frac{L^2}{r^2} = 0$$

and show that it can then be written in the simpler form

$$\left(\frac{du}{d\phi}\right)^2 = R_S u^3 - u^2 + \left(\frac{E}{L}\right)^2$$

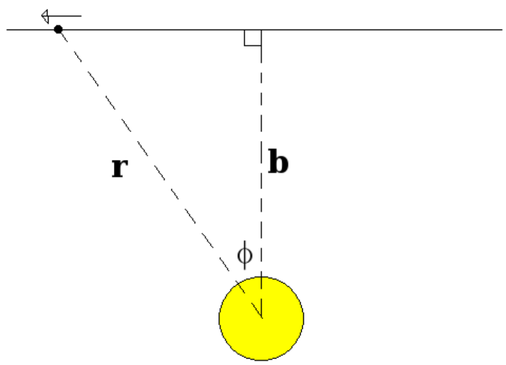

Figure 1: For exercise e). The homogenous equation describes a hypothetical situation where light passes by the Sun in a straight line without changing direction (i.e. a situation where gravity has been "turned off").

d) Differentiate the equation with respect to ϕ (use the chain rule) and show that it can be written as a second-order differential equation:

$$\frac{d^2u}{d\phi^2} + u = ku^2$$

To simplify things, we have introduced the constant $k := \frac{3}{2}R_S$.

Also notice that, according to the laws of physics, $\frac{E}{L}$ is a constant with respect to ϕ, and its derivative vanishes.

Part 2

The purpose of this part is to solve the differential equation

$$\frac{d^2u}{d\phi^2} + u = ku^2$$

Our strategy will be to first find a solution to the homogenous equation

$$\frac{d^2u}{d\phi^2} + u = 0$$

and the find a particular solution to the inhomogenous equation. The final solution will be the sum of the two.

e) First we will look at a case where the path of the light ray does not change at all. If b is the minimum distance from the photon to the center of the star and $r = r(\phi)$ is the distance when the angle is ϕ (see figure 1), show that

$$u = \frac{\cos\phi}{b}$$

and use this to show that u then solves the homogenous equation:

$$\frac{d^2u}{d\phi^2} + u = 0$$

(Hint: b is a constant and does not depend on ϕ).

f) From here on, we will call the solution we found in e) u_h. In order to find a general solution to the (inhomogenous) differential equation, we will also need a particular solution u_s, such that

$$u = u_h + u_s$$

Insert this into the equation

$$\frac{d^2u}{d\phi^2} + u = ku^2$$

and then apply the following approximations :

$$u_s^2 \approx 0 \qquad u_h + 2u_s \approx u_h$$

(Here we presume that the particular solution gives the photon a very small deflection from the straight line represented by u_h, so that the value of u_s is very small and much smaller than u_h):

Show that the inhomogenous equation then can then be written as

$$\frac{d^2u_s}{d\phi^2} + u_s \approx \frac{k}{b^2}\cos^2\phi$$

(Hint: To show this, you will also need u from exercise e), only here renamed to u_h.)

g) To find a solution of the inhomogenous equation we have to make a qualified guess. Looking at the right side of the equation we are led to guess that the solution looks like

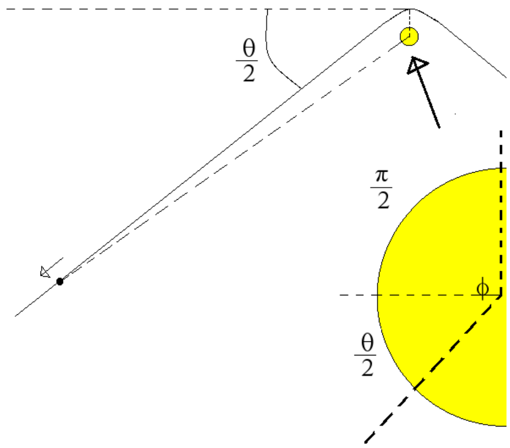

Figure 2: How the angles ϕ and θ are connected. The reason for the choice of $\frac{\theta}{2}$ is, as the figure suggests, that the light is deflected both on its way towards the Sun and away from it. Thus, the *total* angle of deflection becomes $\frac{\theta}{2} + \frac{\theta}{2} = \theta$, and calculating this is our final goal

$$u_s = A\cos^2\phi + B$$

where A and B are constants. Compute the constants A and B and show that the particular solution is given by

$$u_s = \frac{k}{3b^2}\left(2 - \cos^2\phi\right)$$

(Hint: We do not want to have any sinus terms in the solution, but by using a well-known trigomometric rule you should be able to eliminate those terms.)

h) We define the angle of deflection θ (shown in the figure above[2]) so that

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}$$

Use this and the addition formula for cosine to show that

$$\cos\phi \approx -\frac{\theta}{2}$$

[2]Strictly speaking, the relation only applies when the photon is sufficiently far from the Sun that the solid and dotted lines are parallel. This can only happen as the distance between the photon and the Sun approaches infinity (which will indeed happen when we introduce this limit in exercise j). If you don't see a problem with this, there is no need to worry about it - understanding the figure is not important for solving the exercise.

(Hint: The reason we write $\approx$ above is that we have used yet another approximation[3]: We assume that θ is a very small angle, and then

$$\sin\theta \approx \theta$$

You will need this small angle formula to arrive at the correct answer.)

i) If θ is a very small angle, we can also approximate $\theta^2 \approx 0$. Use this approximation, along with the results from exercises e), g) and h) to show that

$$u \approx \frac{2k}{3b^2} - \frac{\theta}{2b}$$

(Hint: Remember that $u = u_h + u_s$, and that you know both solutions from previous exercises. This is a good place to start.)

j) In order to find the angle θ, we now look at the case where the photon is very far away from the Sun, so it will experience no more deflection. In this case $r \to \infty$, which means that $u = \frac{1}{r} \approx 0$. Insert this value for u into the result from i) and show that we get

$$\theta \approx \frac{2 \cdot R_S}{b}$$

after back-substituting the value of k from exercise d).

k) Finally, an exercise with actual numbers! For the Sun, the Schwarzschild radius is $R_S = 3\,\mathrm{km}$. We send a light ray past the Sun so that it only momentarity touches the surface, which means that b is equal to the actual radius of the Sun:

$$b = R_\odot = 696\,000\,\mathrm{km}$$

Use this (and the result from exercise j) to calculate the angle of deflection θ in this case. The answer you get will be in units of radians - calculate what this corresponds to in degrees. Were we correct in making the assumption that the light ray is only slightly deflected (in other words, is θ really a very small angle)?

[3] You can see this by looking at the unit circle for a very small angle and noting that the arc length (in radians) will be measured along an almost vertical line whose length must then be $\sin\theta$.