MAT 1001, Fall 2016

Oblig 1

Deadline: Thursday, September 22, before 1430

You are allowed to work together onthe exercises, but everyone has to handle inn their own solution. The assignment must be handed in in the special box at the 7. floor in Niels Henrik Abels hus (math building) before 14.30, Thursday September 22. Remember to fill in and attach a front page - front pages are found nearby the box, or on the Internet.

Each exercise has a maximum score of 10 points. To be approved you need at least 25 points for the first part, at least 10 points for the second part and at least 30 points for the last part.

The exercise is about a set consisting of three parts, denoted A, B and C. The content of the three parts change stepwise. Throughout the exercises the receipt of the stepwise development will change. Let x_n be the content of A after n step, y_n and z_n the content of B and C after n step, respectively. The dynamic of the system is described by the rule

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = M \cdot \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

where M is the transition matrix which codes the dynamic of the system. The initial state is given by

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

Part 1.

In the first part the dynamic is rather simple, given as follows: The size of A decreases by 20%, the size of B increases by 20%, and the size of C is constant in each step.

a) Determine M in this case.
b) Determine the eigenvalues of M and find the corresponding eigenvectors.

We are interested in what happens to the system when we repeat the stepwise process several times. To get an idea of the development of th system we compute some powers of M.

c) Compute M^2, M^3, M^4 and M^5. What can you say about M^n when $n \to \infty$ in this case?
d) Find an equilibrium state for the process, i.e. a state (x_s, y_s, z_s) such that

$$M \cdot \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}$$

Let the **attractor basin** for the equilibrium state (x_s, y_s, z_s) be the set of all states (x, y, z) such that

$$M^n \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} \quad \text{when} \quad n \to \infty$$

e) Determine the attractor basin for the equilibrium state (x_s, y_s, z_s) you found in exercise d).

Part 2.

In part 1 it was rather obvious that the size of A would decrease during the process, that the size of B would increase and that the size of C would remain constant. In this part we shall modify the dynamic to change this behavior. We do this by changing the dynamic such that in each step 10% of the size of C is moved from B to A. This gives a new transition matrix

$$M = \begin{pmatrix} 0.8 & 0 & 0.1 \\ 0 & 1.2 & -0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

f) Determine the eigenvalues and the corresponding eigenvectors of M in this case.
g) Find an equilibrium state and the corresponding attractor basin for this M.

Part 3.

In parts 1 and 2 the transition matrices were both (upper) triangular, in part 1 even diagonal. As you maybe experienced, it was rather easy to compute the eigenvalues and even the eigenvectors in that case. In the last part we shall modify the dynamic further, obtaining a more complicated transition matrix. This is done by adding the following moves to the system, in each step we decrease the size of B and C, by 10% of the size of A. At the same time we increase the size of A and C, by 10% of the size of B. The transition matrix now look like

$$M = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ -0.1 & 1.2 & -0.1 \\ -0.1 & 0.1 & 1 \end{pmatrix}$$

h) Compute the characteristic polynomial of M and use it to show that the eigenvalues of M now will be $\lambda_1 = 1$, $\lambda_2 = 1.1$ and $\lambda_3 = 0.9$
i) Compute the eigenvectors corresponding to the eigenvalues given in exercise h).

Let the initial state be

$$s_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 5 \end{pmatrix}$$

j) Write the initial state s_0 as a linear combination of the eigenvectors you found in exercise i).
k) Use your answer in k) to determine $P^n \cdot s_0$, as a function of n.
l) If your computation is correct you will see that $P^n \cdot s_0$ converges to a certain state when $n \to \infty$. Determine this state.
m) If we modify the initial state s_0' slightly, then $P^n \cdot s_0'$ will no more converge to an equilibrium state. Which modification to you think we have in mind?

END.