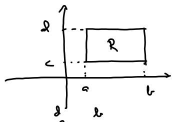
Dobbeltintegraler

Forrige gang:

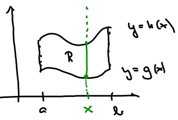


$$\iint_R f(x,y)\,dA$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

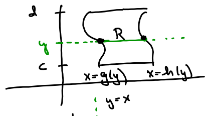
Integrasjon over mer generelle områder:

Type 1



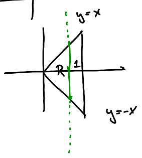
$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$

Type 2:



$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,y)\,dx \right] dy$$

Eksempel



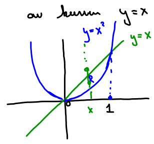
$f(x,y) = x+y$

$$\iint (x+y)\,dA = \int_0^1 \left[\int_{-x}^{x} (x+y)\,dy \right] dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=-x}^{y=x} dy = \int_0^1 \left(\left[x^2 + \cancel{\frac{x^2}{2}} \right] - \left[-x^2 + \cancel{\frac{(-x)^2}{2}} \right] \right) dx$$

$$= \int_0^1 2x^2\,dx = \left[2\frac{x^3}{3} \right]_0^1 = \frac{2}{3} - 0 = \underline{\underline{\frac{2}{3}}}$$

Eksempel: Regn ut $\iint_R x^2 y\,dA$ der R er området avgrenset av kurvene $y = x$ og $y = x^2$.



Finner skjæringspunktene: $x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Downarrow$

$x = 0 \quad x = 1$

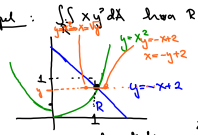
$$\iint_R x^2 y\,dA = \int_0^1 \left[\int_{x^2}^{x} x^2 y\,dy \right] dx$$

$$= \int_0^1 \left[x^2 \frac{y^2}{2} \right]_{y=x^2}^{y=x} dx = \int_0^1 \left[\frac{1}{2} x^2 x^2 - \frac{1}{2} x^2 \underbrace{(x^2)^2}_{x^4} \right] dx$$

$$= \frac{1}{2} \int_0^1 (x^4 - x^6)\,dx = \frac{1}{2} \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{5} - \frac{1}{7} \right] = \frac{1}{\cancel{2}} \left[\frac{\cancel{7-5}}{35} \right] = \underline{\underline{\frac{1}{35}}}$$

Eksempel: $\iint xy^2\,dA$ hvor R er gitt ved:



R er området under begge kurvene $y = x^2$ og $y = -x+2$

Finner skjæringspunktet:

$x^2 = -x+2 \Rightarrow x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-2)}}{2} = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$= \frac{-1 \pm 3}{2} = \begin{cases} 1 \leftarrow \text{denne må det være.} \\ -2 \end{cases}$$

Velger å integrere i x-retning først:

$$\int_0^1 \left[\int_{\sqrt{y}}^{-y+2} xy^2\,dx \right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 \right]_{x=\sqrt{y}}^{x=-y+2} dy = \int_0^1 \left[\frac{(-y+2)^2}{2} y^2 - \frac{(\sqrt{y})^2}{2} y^2 \right] dy$$

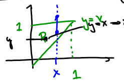
$$= \frac{1}{2} \int_0^1 \left[(y^2 + 2(-y)\cdot 2 + 2^2) y^2 - y^3 \right] dy$$

$$= \frac{1}{2} \int_0^1 \left[y^4 - 4y^3 + 4y^2 - y^3 \right] dy = \frac{1}{2} \int_0^1 (y^4 - 5y^3 + 4y^2)\,dy$$

$$= \frac{1}{2} \left[\frac{y^5}{5} - 5\frac{y^4}{4} + 4\frac{y^3}{3} \right]_0^1 = \frac{1}{2} \left[\frac{1}{5} - \frac{5}{4} + \frac{4}{3} \right]$$

$= \ldots\ldots\ldots$

Eksempel: Integrer $f(x,y) = e^{y^2}$ over området



$y = x \Rightarrow x = y$ Prøver: $\iint_R e^{y^2}\,dy = \underbrace{\int_0^1 \left[\int_x^1 e^{y^2}\,dy \right] dx}_{\text{Får ikke til.}}$

Prøver isteden:

$$\iint_R e^{y^2}\,dy = \int_0^1 \left[\int_0^y e^{y^2}\,dx \right] dy$$

$\boxed{u = y^2 \quad du = 2y\,dy}$

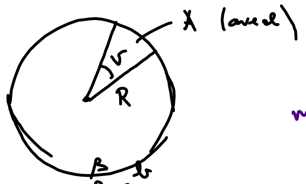
$$= \int_0^1 \left[e^{y^2} \cdot x \right]_{x=0}^{x=y} dy = \int_0^1 \left[e^{y^2} y \right] dy$$

$$= \left[\frac{1}{2} e^{y^2} \right]_0^1 = \frac{1}{2} e^1 - \frac{1}{2} e^0 = \frac{1}{2} e - \frac{1}{2}$$

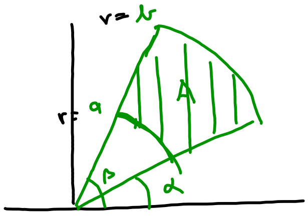
$$= \underline{\frac{1}{2}(e-1)}$$

Dobbeltintegraler i polarkoordinater

Arealet til en sirkelsektor:



$$\frac{A}{\pi R^2} = \frac{v}{2\pi} \Rightarrow A = \frac{R^2 v}{2} = \frac{1}{2}R^2 v$$

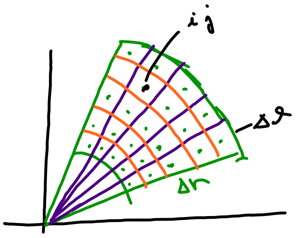


mystisk

$$\iint_A f(x,y)\,dA = \int_\alpha^\beta \left[\int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\right] d\theta$$

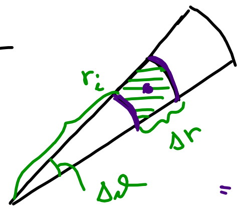
mystisk

$$= \int_a^b \left[\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\right] dr$$



$$V = \sum V_{ij} \approx \sum f(x_{ij}, y_{ij})\, A_{ij}$$

Hva er A_{ij}: Differansen mellom to sirkelsegmenter:



$$\frac{1}{2}(r_i + \Delta r)^2 \Delta\theta - \frac{1}{2}(r_i)^2 \Delta\theta$$

$$= \frac{1}{2}\left[\cancel{r_i^2} + 2r_i\Delta r + \Delta r^2\right]\Delta\theta - \frac{1}{2}\cancel{r_i^2}\,\Delta\theta$$

$$= \left(r_i\,\Delta r + \frac{\Delta r^2}{2}\right)\Delta\theta$$

$$= \underbrace{\left(r_i + \frac{\Delta r}{2}\right)}_{\text{radien i midtpunktet.}} \Delta r\,\Delta\theta$$

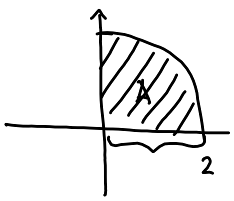
Dette gir

$$V \approx \sum f(x_{ij}, y_{ij})\, A_{ij}$$

$$= \underbrace{\sum f(r_i\cos\theta_j, r_i\sin\theta_j)\, r_i}\, \Delta r\, \Delta\theta$$

$$\to \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Eksempel: $\iint_A xy^2\,dA$ over kvartsirkelen A



$$\iint_A xy^2\,dA = \int_0^{\pi/2}\left[\int_0^2 r\cos\theta\,(r\sin\theta)^2 \cdot r\,dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^2 r^4 \sin^2\theta\cos\theta\,dr\right]d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^5}{5}\sin^2\theta\cos\theta\right]_{r=0}^{r=2}$$

$$= \int_0^{\pi/2}\left[\frac{32}{5}\sin^2\theta\cos\theta\right]d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{32}{5}\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}$$

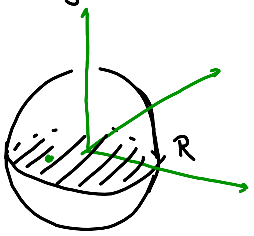
$$= \frac{32}{5}\left[\frac{1}{3} - \frac{0}{3}\right] = \frac{32}{15}$$

Eksempel: Regn ut volumet til en kule med radius R:

$$x^2+y^2+z^2=R^2 \Rightarrow z=\pm\sqrt{R^2-x^2-y^2}$$

$x = r\cos\theta$

$y = r\sin\theta$



$$V = 2\iint_A \underbrace{\sqrt{R^2-x^2-y^2}}_{R^2-(\underbrace{x^2+y^2}_{r^2})}\, dA$$

A ← Sirkel med radius R.

$$= 2\int_0^R \left[\int_0^{2\pi} \sqrt{R^2-r^2}\, r\, d\theta\right] dr$$

$$= 2\int_0^R \left[\sqrt{R^2-r^2}\cdot r\theta\right]_{\theta=0}^{\theta=2\pi} dr = 2\int_0^R 2\pi\sqrt{R^2-r^2}\, r\, dr$$

$$= 4\pi\int_0^R \sqrt{\underbrace{R^2-r^2}_{u}}\cdot r\, dr$$

$u = R^2-r^2$

$du = -2r\,dr$

$r\,dr = -\frac{1}{2}\,du$

Mellomregning: Regner ut $\int\sqrt{R^2-r^2}\, r\, dr$

$$= \int\sqrt{u}\left(-\tfrac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right] + C = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(R^2-r^2)^{3/2} + C$$

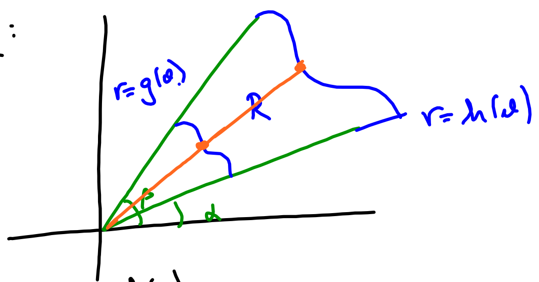
Går tilbake til volumberegningen:

$$V = 4\pi\int_0^R \sqrt{R^2-r^2}\, r\, dr = 4\pi\left[-\frac{1}{3}(R^2-r^2)^{3/2}\right]_{r=0}^{r=R}$$

$$= -\frac{4\pi}{3}\Big[(\overbrace{R^2-R^2}^{0})^{3/2} - (\overbrace{R^2-0^2}^{R^2})^{3/2}\Big] = -\frac{4\pi}{3}(-R^3)$$

$$= \frac{4\pi}{3}R^3$$

Mer generelt:



$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \left[\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\right] d\theta$$