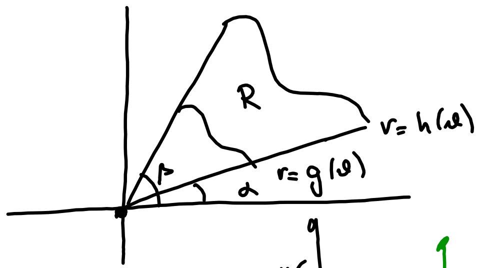
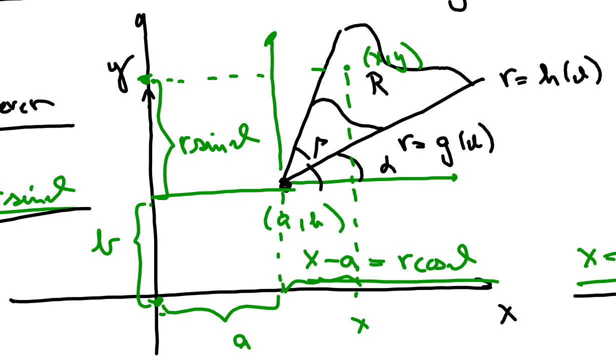
# Integrasjon i polarkoordinater




$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \left[\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\right] d\theta$$

**Generaliser**

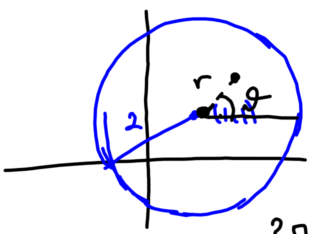
$y = b + r\sin\theta$




$x = a + r\cos\theta$

**Vi får:**
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\left[\int_{g(\theta)}^{h(\theta)} f(a+r\cos\theta,\ b+r\sin\theta)\, r\,dr\right]d\theta$$

**Eksempel:** Regn ut $\iint_R xy\,dA$ der R er sirkelen om $(1,1)$ med radius 2.




$$\iint_A xy\,dA = \int_0^{2\pi}\left[\int_0^2 \underbrace{(1+r\cos\theta)}_{x}\,\underbrace{(1+r\sin\theta)}_{y}\, r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\int_0^2 (1 + r\sin\theta + r\cos\theta + r^2\cos\theta\sin\theta)\, r\; dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\int_0^2 (r + r^2\sin\theta + r^2\cos\theta + r^3\cos\theta\sin\theta)\, dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2} + \frac{r^3}{3}\sin\theta + \frac{r^3}{3}\cos\theta + \frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^{r=2} d\theta$$

$u = \sin\theta$

$$= \int_0^{2\pi}\left[2 + \frac{8}{3}\sin\theta + \frac{8}{3}\cos\theta + 4\cos\theta\sin\theta - 0\right] d\theta$$

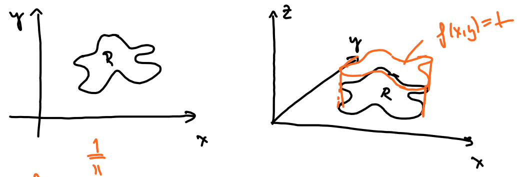
$$= \left[2\theta - \frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta + 2\sin^2\theta\right]_0^{2\pi}$$

$$= 2\cdot 2\pi - \underbrace{0} = \underline{\underline{4\pi}}$$

øvre og nedre grense er like siden cos og sin har periode $2\pi$

## Arealer og tyngdepunkter

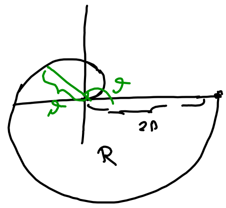
**Areal:**




$$\iint_R \overset{\overset{1}{\|}}{f}(x,y)\,dA = \text{areal}(R)\cdot 1 = \text{areal}(R)$$

$$\text{Areal}(R) = \iint_R 1\,dA.$$

**Eksempel:** Spiral: $r = \theta$



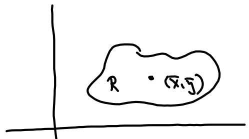

$$\text{areal}(R) = \iint_R 1\,dA$$

$$= \int_0^{2\pi}\Big[\int_0^{\theta} 1\cdot r\,dr\Big]d\theta$$

$$= \int_0^{2\pi}\Big[\frac{r^2}{2}\Big]_0^{\theta}\,d\theta = \int_0^{2\pi}\frac{\theta^2}{2}\,d\theta = \Big[\frac{\theta^3}{6}\Big]_0^{2\pi}$$

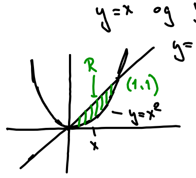
$$= \frac{(2\pi)^3}{6} = \frac{8\pi^3}{6} = \underline{\underline{\frac{4}{3}\pi^3}}$$

**Tyngdepunkt:**




$$\bar{x} = \frac{\iint_R x\,dA}{\text{areal}(R)}, \quad \bar{y} = \frac{\iint_R y\,dA}{\text{areal}(R)}$$

**Eksempel:** Regn ut tyngdepunktet til området avgrenset av $y = x$ og $y = x^2$:




Regner først ut arealet:

$$\text{areal}(R) = \iint_R 1\,dA = \int_0^1\Big[\int_{x^2}^{x} 1\,dy\Big]dx$$

$$= \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 [x - x^2]\,dx$$

$$= \Big[\frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \underline{\underline{\frac{1}{6}}}$$

**Regner ut:**

$$\iint_R x\,dA = \int_0^1\Big[\int_{x^2}^{x} x\,dy\Big]dx$$

$$= \int_0^1 [xy]_{y=x^2}^{x}\,dx = \int_0^1 [x\cdot x - x\cdot x^2]\,dx$$

$$= \int_0^1 [x^2 - x^3]\,dx = \Big[\frac{x^3}{3} - \frac{x^4}{4}\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \underline{\frac{1}{12}}.$$

**x-koordinat til tyngdepunktet:**

$$\bar{x} = \frac{\iint_R x\,dA}{\text{areal}(R)} = \frac{\frac{1}{12}}{\frac{1}{6}} = \underline{\underline{\frac{1}{2}}}$$

**Regner ut:**

$$\iint_R y\,dA = \int_0^1\Big[\int_{x^2}^{x} y\,dy\Big]dx = \int_0^1 \Big[\frac{y^2}{2}\Big]_{y=x^2}^{y=x}\,dx$$

$$= \int_0^1\Big[\frac{x^2}{2} - \frac{(x^2)^2}{2}\Big]dx = \int_0^1\Big[\frac{x^2}{2} - \frac{x^4}{2}\Big]dx$$

$$= \Big[\frac{x^3}{6} - \frac{x^5}{10}\Big]_0^1 = \frac{1}{6} - \frac{1}{10} = \frac{5}{30} - \frac{3}{30} = \frac{2}{30} = \underline{\underline{\frac{1}{15}}}$$

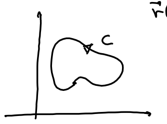
**y-koordinaten til tyngdepunktet:**

$$\bar{y} = \frac{\iint_R y\,dA}{\text{areal}(R)} = \frac{\frac{1}{15}\cdot 6}{\frac{1}{6}\cdot 6} = \frac{6}{15} = \underline{\underline{\frac{2}{5}}}$$

**Tyngdepunkt:** $\underline{(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)}.$

## Greens teorem

**Linje integral:**




$$\vec F(x,y) = (p(x,y),\, q(x,y))$$

$$\vec r(t) = (x(t),\, y(t)) \qquad a \le t \le b.$$

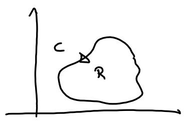
$$\int_C \vec F \cdot d\vec r = \int_a^b (p(x(t),y(t)),\, q(x(t),y(t))) \cdot (x'(t),\, y'(t))\, dt$$

$$= \int [p(x(t),y(t)) \cdot x'(t) + q(x(t),y(t))\, y'(t)]\, dt$$

$$= \int_a^b p(x(t),y(t))\, \underbrace{x'(t)\,dt}_{dx} + q(x(t),y(t))\, \underbrace{y'(t)\,dt}_{dy}$$

$$\overset{\text{Notasjon}}{=} \int_C p\, dx + q\, dy$$

**Greens teorem**




R et område i planet, sammenhengende og uten hull(er).

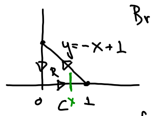
C er randkurven til R, orientert mot klokken.

Da gjelder:

$$\oint_C p\, dx + q\, dy = \iint_R \overbrace{\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)}^{\text{curl } \vec F} dA$$

**Eksempel:** Regn ut $\int_C p\,dx + q\,dy$ når $\vec F(x,y) = (\underbrace{xy}_{p(x,y)},\, \underbrace{x+y}_{q(x,y)})$ og C er trekanten med hjørner i $(0,0),\ (1,0),\ (0,1)$

Bruker Greens teorem:




$$\int_C p\,dx + q\,dy = \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dA$$

$$= \iint_R (1 - x)\, dA = \int_0^1 \left[\int_0^{-x+1} (1-x)\, dy\right] dx$$

$$= \int_0^1 \left[(1-x)\, y\right]_{y=0}^{y=(1-x)} dx = \int_0^1 (1-x)^2\, dx$$

$$= \int_0^1 (x-1)^2\, dx \quad \left[\frac{(x-1)^3}{3}\right]_0^1 = \left[0 - \left(\frac{(-1)^3}{3}\right)\right] = \underline{\underline{\frac{1}{3}}}$$

**Greens teorem til å regne ut arealer:**

$$\int_C 0\, dx + x\, dy = \iint_R (1 - 0)\, dA = \iint_R 1\, dA = \text{areal}(R)$$

$p = 0,\ q = x$

$\frac{\partial q}{\partial x} = 1$

$\frac{\partial p}{\partial y} = 0$

**Altså:** $\text{areal}(R) = \int_C x\, dy$

Tilsvarende: $$\int_C y\, dx + 0\, dy = \iint_R (0 - 1)\, dA = -\iint_R 1\, dA = -\text{areal}(R)$$

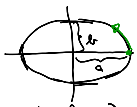
$p = y,\ q = 0$

$\frac{\partial p}{\partial y} = 1,\ \frac{\partial q}{\partial x} = 0$

$$\text{areal}(R) = -\int_C y\, dx$$

**Oppsummering:** $\text{areal}(R) = \int_C x\, dy = -\int_C y\, dx = \frac{1}{2}\left(\int_C x\, dy - \int_C y\, dx\right)$

**Eksempel:** Areal til en ellipse.




$$\vec r(t) = (a\cos t,\, b\sin t), \quad 0 \le t \le 2\pi$$

**Areal:** $\text{areal}(R) = \int_C x\, dy$ $\qquad \vec r\,'(t) = (\underbrace{-a\sin t}_{\text{"}dx\text{"}},\, \underbrace{b\cos t}_{\text{"}dy\text{"}})$

$$= \int_0^{2\pi} (a\cos t)(b\cos t)\, dt = ab \int_0^{2\pi} \cos^2 t\, dt$$

$\cos 2t = 2\cos^2 t - 1$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$$= ab \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = \frac{ab}{2} \int_0^{2\pi} (1 + \cos 2t)\, dt$$

$$= \frac{ab}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{ab}{\cancel{2}}\left[\left(\cancel{2}\pi + \underbrace{\frac{\sin 4\pi}{2}}_{=0}\right) - \left(0 + \underbrace{\frac{\sin 0}{2}}_{0}\right)\right] = \underline{\underline{\pi ab}}$$

**Variant:**

$$\text{areal} = \frac{1}{2} \int_C x\, dy - y\, dx$$

$$\vec r(t) = (a\cos t,\, b\sin t)$$

$$\vec r\,'(t) = (-a\sin t,\, b\cos t)$$

$$= \frac{1}{2} \int_0^{2\pi} (a\cos t\, b\cos t - b\sin t\,(-a\sin t))\, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = \frac{ab}{2} \int_0^{2\pi} 1\, dt = \frac{ab}{\cancel{2}} \cdot \cancel{2}\pi = \underline{\underline{\pi ab}}$$