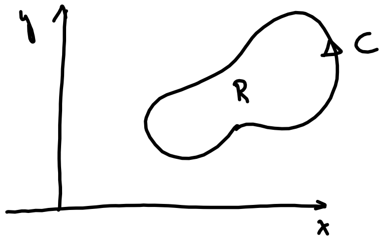
Greens teorem



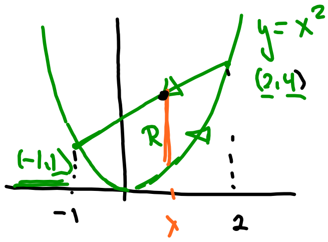
$$\vec{F}(x,y) = (p(x,y), q(x,y))$$

$$\int_C p\,dx + q\,dy = \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dA$$

Alternativ skrivemåte:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\; dA$$

Eksempel: $\displaystyle\int_C \overbrace{(x^2y + xe^x)}^{p}\,dx + \overbrace{(xy^3 + e^{\sin y})}^{q}\,dy$



Formel for rett linje.

$$y-1 = \frac{4-1}{2-(-1)}(x-(-1))$$

$$y-1 = (x+1) \Rightarrow y = x+2$$

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$$\iint_R \left(\frac{\partial}{\partial x}(xy^3 + e^{\sin y}) - \frac{\partial}{\partial y}(x^2y + xe^x)\right) dA$$

$$= \iint_R (y^3 - x^2)\,dA = \int_{-1}^{2}\left[\int_{x^2}^{x+2} (y^3 - x)\,dy\right] dx$$

$$= \int_{-1}^{2}\left[\frac{y^4}{4} - xy\right]_{y=x^2}^{y=(x+2)} dx = \int_{-1}^{2}\left(\frac{(x+2)^4}{4} - x(x+2)\right) - \left(\frac{(x^2)^4}{4} - x\cdot x^2\right) dx$$

$$= \int_{-1}^{2}\left[\frac{1}{4}(x+2)^4 - x^2 - 2x + \frac{1}{4}x^8 + x^4\right] dx$$

$$= \left[\frac{1}{4}\frac{(x+2)^5}{5} - \frac{x^3}{3} - x^2 + \frac{1}{4}\frac{x^9}{9} + \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \left[\frac{1}{4}\frac{4^5}{5} - \frac{8}{3} - 4 + \frac{1}{4}\frac{2^9}{9} + \frac{2^5}{5}\right]$$

$$- \left[\frac{1}{4}\cdot\frac{1}{5} + \frac{1}{3} - 1 - \frac{1}{4}\frac{1}{9} - \frac{1}{5}\right]$$

$$= \left[\frac{4^5-1}{20} - 3 + \frac{2^9+1}{36} + \frac{2^5+1}{5} - 4 + 1\right]$$

$$= \ldots$$

Sammenheng mellom Greens teorem og konservative felt:

Anta at $\vec{F}$ er konservativt, da er $\text{curl}\,\vec{F} = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = \underline{0}$.

Hvis vi integrerer $\vec{F}$ rundt en lukket kurve C:

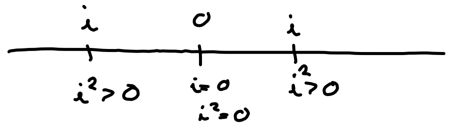
$$\int_C \vec{F}\cdot d\vec{r} = \underline{0}$$

Greens teorem: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\;dA = \iint_R 0\,dA = \underline{0}$

Komplekse tall

Hva er kvadratroten av -1? Finnes det en slik kvadratrot? $i^2=-1$

Ikke på tallinjen:



Anta at det finnes en kvadratrot i av -1 (dvs $i^2=-1$) som oppfører seg som vanlige tall i regninger. Hvilke konsekvenser får dette?

$z=a+bi$, der $a,b\in\mathbb{R}$ kalles et komplekst tall

Eksempel: $z=3+17i$, $z=\sqrt{2}+\pi i$, $z=-\frac{3}{4}+12i$

Addisjon: $z=a+ib$, $w=c+id$

$$z+w=a+ib+c+id=(a+c)+(b+d)i$$

Eksempel: $z=3+4i$, $w=-2+3i$

$$z+w=3+4i+(-2)+3i=\underline{1+7i}$$

Subtraksjon: $z=a+ib$, $w=c+id$

$$z-w=(a+ib)-(c+id)=a+ib-c-id=(a-c)+(b-d)i$$

Eksempel: $z=2-i$, $w=7+5i$

$$z-w=2-i-(7+5i)=2-i-7-5i=-5-6i=(-5)+(-6)i$$

Multiplikasjon: $z=a+ib$, $w=c+id$

$$z\cdot w=(a+ib)(c+id)=ac+iad+ibc+\underbrace{i^2}_{-1}bd$$

$$=ac+iad+ibc-bd=(ac-bd)+i(ad+bc)$$

Eksempel: $z=3-4i$, $w=-2+3i$

$$zw=(3-4i)(-2+3i)=-6+9i+8i+(-4i)(3i)$$

$$=-6+17i\underbrace{-\underbrace{i^2}_{-1}\cdot 12}_{+12}=\underline{\underline{6+17i}}$$

Divisjon: $z=a+ib$, $w=c+id$

$$\frac{z}{w}=\frac{a+ib}{c+id}=\frac{(a+ib)\cdot(c-id)}{(c+id)\cdot(c-id)}=\frac{ac-iad+ibc\overbrace{-i^2bd}^{+bd}}{c^2-\cancel{icd}+\cancel{idc}\underbrace{-i^2d}_{d^2}}$$

$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}$$

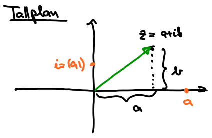
Eksempel: $z=2+4i$, $w=3-2i$

$$\frac{z}{w}=\frac{2+4i}{3-2i}=\frac{(2+4i)(3+2i)}{(3-2i)(3+2i)}=\frac{6+4i+12i\overbrace{-8i^2}^{8}}{3^2\underbrace{-2^2i^2}_{+4}}$$

$$=\frac{14+16i}{13}=\frac{14}{13}+\frac{16}{13}i$$

Geometrisk tolkning

Tallplan

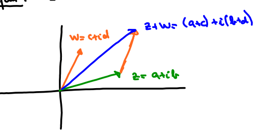


$z = a+ib$

a et reell : $a = a+0i$

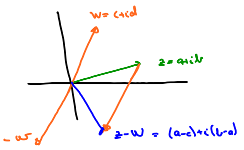
$i = 0+1\cdot i$

Addisjon: $z = a+ib$, $w = c+id$



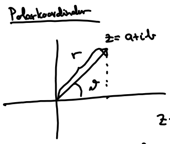
Addisjon av komplekse tall svarer til addisjon av vektorer.

Tilsvarende for subtraksjon



Hva skjer med multiplikasjon geometrisk?

Polarkoordinater



$r = \sqrt{a^2+b^2}$

$a = r\cos\vartheta$

$b = r\sin\vartheta$

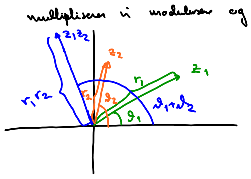
r kalles modulus eller lengden til z

$r = \sqrt{a^2+b^2} = |z|$

ϑ kalles argumentet til z.

$$z = a+ib = r\cos\vartheta + i r\sin\vartheta = r(\cos\vartheta + i\sin\vartheta)$$

Caspar Wessels oppdagelse: Når vi ganger sammen to komplekse tall, multipliserer vi modulusene og adderer argumentene.



z_1z_2

Husk:

$\cos(u+v) = \cos u\cos v - \sin u\sin v$

$\sin(u+v) = \cos u\sin v + \sin u\cos v$

$z_1 = a_1+b_1 i = r_1\cos\vartheta_1 + i r_1\sin\vartheta_1$

$z_2 = a_2+b_2 i = r_2\cos\vartheta_2 + i r_2\sin\vartheta_2$

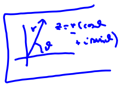
$$z_1z_2 = (r_1\cos\vartheta_1 + i r_1\sin\vartheta_1)(r_2\cos\vartheta_2 + i r_2\sin\vartheta_2)$$

$$= r_1r_2\cos\vartheta_1\cos\vartheta_2 + i r_1r_2\cos\vartheta_1\sin\vartheta_2$$
$$+ i r_1r_2\sin\vartheta_1\cos\vartheta_2 - r_1r_2\sin\vartheta_1\sin\vartheta_2$$

$$= r_1r_2\big[\underbrace{(\cos\vartheta_1\cos\vartheta_2 - \sin\vartheta_1\sin\vartheta_2)}_{\cos(\vartheta_1+\vartheta_2)} + i\underbrace{(\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2)}_{\sin(\vartheta_1+\vartheta_2)}\big]$$

$$= \underbrace{\overbrace{r_1r_2}^{r}[\cos(\overbrace{\vartheta_1+\vartheta_2}^{\vartheta}) + i\sin(\overbrace{\vartheta_1+\vartheta_2}^{\vartheta})]}$$

= komplekst tall med modulus r_1r_2 og argument $\vartheta_1+\vartheta_2$.



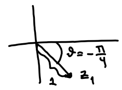
Forkortet skrivemåte: $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

$z_1 = r_1(\cos\vartheta_1 + i\sin\vartheta_1) = r_1e^{i\vartheta_1}$, $z_2 = r_2e^{i\vartheta_2}$

$$z_1\cdot z_2 = r_1e^{i\vartheta_1}\cdot r_2e^{i\vartheta_2} = r_1r_2\,e^{i\vartheta_1+i\vartheta_2} = r_1r_2\,e^{i(\vartheta_1+\vartheta_2)}$$

Eksempel: $z_1 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$, $r_1 = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2}$

$= \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

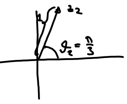


$z_1 = re^{i\vartheta} = e^{-i\frac{\pi}{4}}$

$z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$r_2 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$



$z_2 = r_2e^{i\vartheta_2} = e^{i\frac{\pi}{3}}$

$\cos\vartheta_2 = \frac{a}{r} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

$\vartheta_2 = \frac{\pi}{3}$

Regner ut produktet på to måter:

$$z_1z_2 = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{4} - i\frac{\sqrt{2}}{4}\underbrace{-i^2\frac{\sqrt{6}}{4}}_{+\frac{\sqrt{6}}{4}}$$

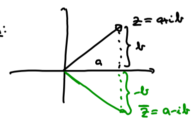
$$= \left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right) + i\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)$$

$$z_1z_2 = e^{-i\frac{\pi}{4}}\cdot e^{i\frac{\pi}{3}} = e^{i\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = e^{i\left(\frac{4\pi}{12}-\frac{3\pi}{12}\right)} = e^{i\frac{\pi}{12}} = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$$

Altså: $\cos\frac{\pi}{12} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$

$\sin\frac{\pi}{12} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$

Konjugasjon:



Hvis $z = a+ib$ er et komplekst tall, så kalles $\bar{z} = a-ib$ den konjugerte til z.

Regneregler: (i) $\overline{z_1+z_2+\cdots+z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$

(ii) $\overline{z_1z_2\cdots z_n} = \bar{z}_1\bar{z}_2\cdots\bar{z}_n$

Viser (ii) for to faktorer: $z_1 = a+ib$, $z_2 = c+id$, skal vise at

$\overline{z_1z_2} = \bar{z}_1\cdot\bar{z}_2$

$$\overline{z_1z_2} = \overline{(a+ib)(c+id)} = \overline{ac+iad+ibc-bd} = \overline{(ac-bd)+i(ad+bc)}$$
$$= (ac-bd) - i(ad+bc)$$

$$\bar{z}_1\cdot\bar{z}_2 = (a-ib)(c-id) = ac - iad - ibc \underbrace{+i^2bd}_{-bd} = (ac-bd) - i(ad+bc)$$