Prøveeksamen 1

Opg. 1

a) $A = \begin{pmatrix} 0.3 & 0.6 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.2 & 0 & 0.8 \end{pmatrix}$

$1 \quad 1 \quad 1$

b) $\chi_A(\lambda) = \det\left(A - \lambda I\right)$ $\quad \lambda_1 = 1$

$\lambda_2 = \frac{7}{10}$

$= \det\begin{pmatrix} 0.3-\lambda & 0.6 & 0.1 \\ 0.5 & 0.4-\lambda & 0.1 \\ 0.2 & 0 & 0.8-\lambda \end{pmatrix}$ $\quad \lambda_3 = -\frac{1}{5}$

$(0.3-\lambda)\cdot\left((0.4-\lambda)\cdot(0.8-\lambda)\right)$

$-0.6\cdot\left(0.5\cdot(0.8-\lambda) - 0.1\cdot 0.2\right)$

$+\ 0.1\left(0 - 0.2\cdot(0.4-\lambda)\right)$

$$\chi_A(\lambda) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_1$$

c) $$10 \cdot \begin{pmatrix} -0.7 & 0.6 & 0.1 \\ 0.5 & -0.6 & 0.1 \\ 0.2 & 0 & -0.2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 10 \cdot \vec{0}$$

$$\begin{array}{ll} \mathrm{I} & -7x + 6y + z = 0 \\ \mathrm{II} & 5x - 6y + z = 0 \\ \mathrm{III} & 2x \qquad\quad -2z = 0 \end{array}$$

$$\begin{array}{ll} \mathrm{I} & -6x + 6y = 0 \\ \mathrm{II} & 6x - 6y = 0 \end{array} \qquad \begin{pmatrix} x \\ x \\ x \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathrm{II} = -\mathrm{I}$$

$$x = y$$

d) $\lambda_1 = 1 \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = \frac{7}{10} \quad v_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$\lambda_3 = -\frac{1}{5} \quad v_3 = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$

Basis

$\{v_1, \dots, v_n\}$ lin. unab

$\text{Span}\{v_1, \dots, v_n\} = V$

$$\begin{pmatrix} 1 & -1 & -5 \\ 1 & -1 & 4 \\ 1 & 2 & 1 \end{pmatrix} = 1 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + (-5) \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= -9 - 3 - 15 = -27 \neq 0$$

e) $\chi_A = \det(A - \lambda I)$

$\chi_{A^T}(\lambda) = \det(A^T - \lambda I)$

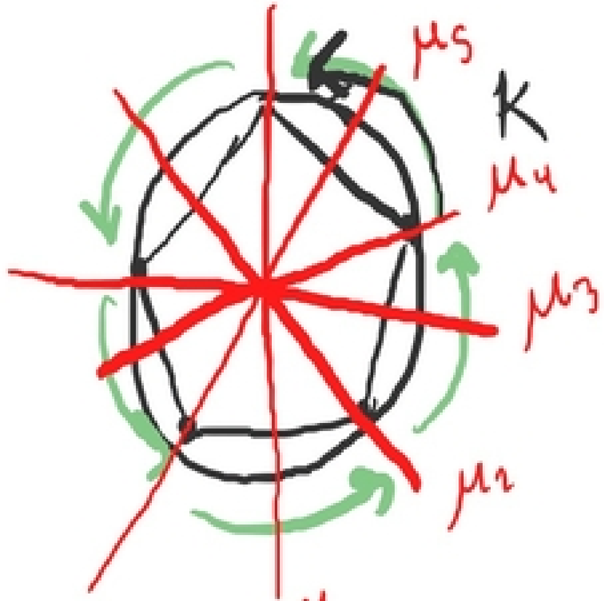
Prop 2.5.6 A^T og A har samme egenver.

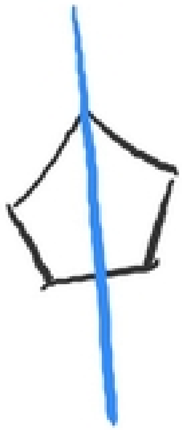
Prop 2.3.8 $\det(A) = \det(A^T)$ $(A^T)^T$

$\chi_{A^T}(\lambda) = \det(A^T - \lambda I) = \det((A^T - \lambda I)^T)$

$= \det(A - \lambda I) = \chi_A(\lambda)$

2)





a) $\frac{2\pi}{5}$

$\rho^n = Id$

$\rho^5 \leftrightarrow 5 \cdot \frac{2\pi}{5} = 2\pi \leftrightarrow Id$

b) $\langle \rho, \mu_1 \rangle$

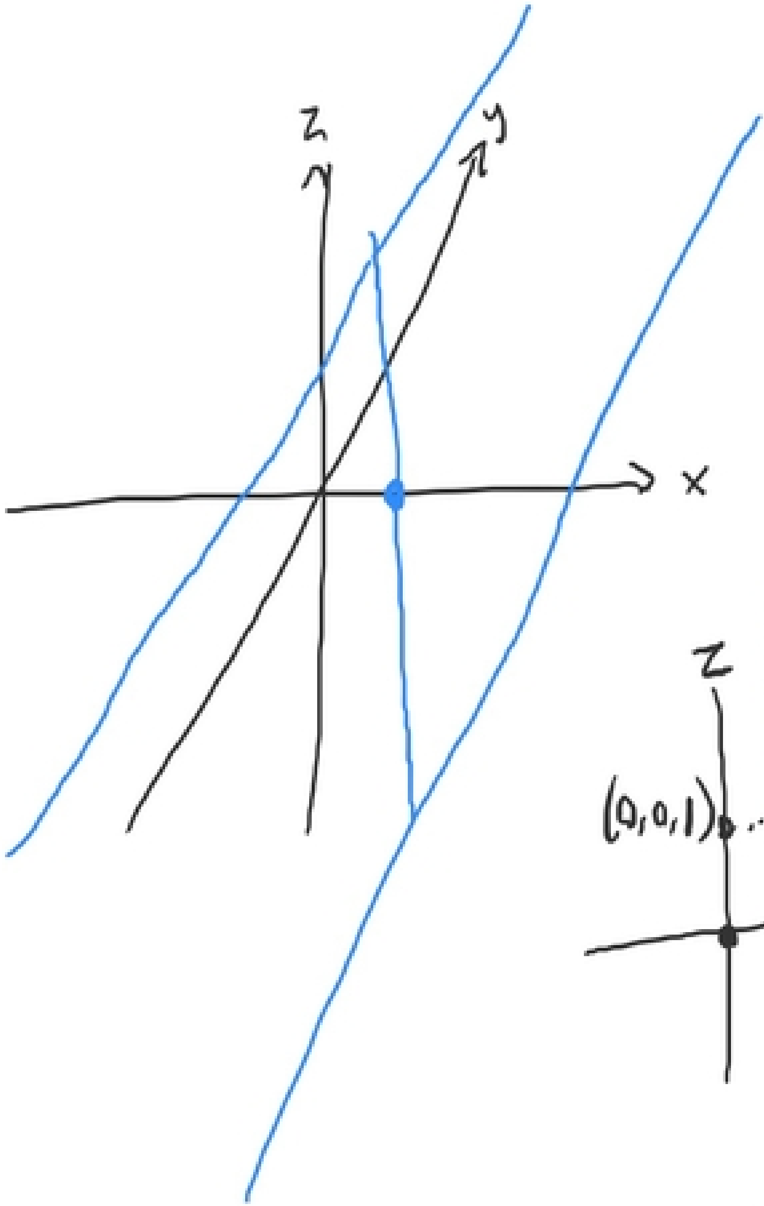
$\mu_2 = \rho \circ \mu_1$

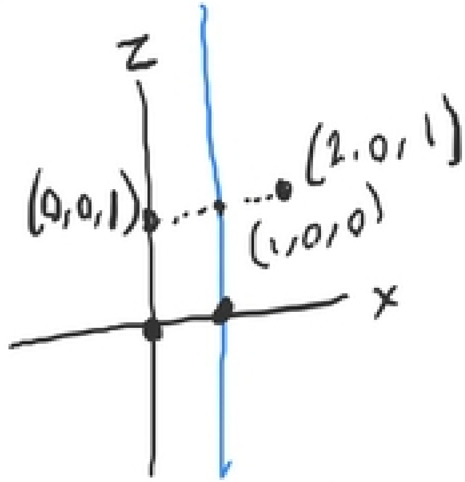
$\mu_3 = \rho^2 \circ \mu_1$

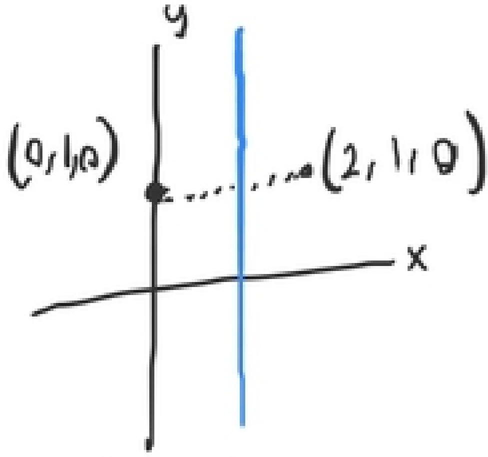
$\mu_4 = \rho^3 \circ \mu_1$

$\mu_5 = \rho^4 \mu_1$

3) R







$$R(\vec{x}) = A x + \vec{b}$$

$$R(\vec{0}) = A \cdot \vec{0} + \vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$R\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad R\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$R\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$R\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = A \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = A \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$$

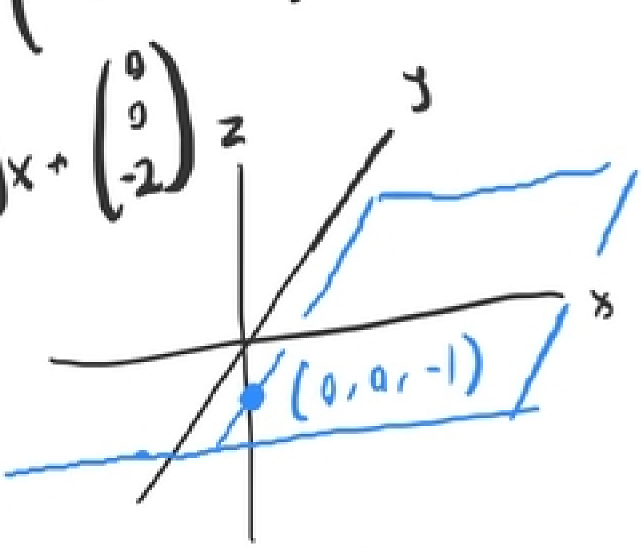
$$S_1 = \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

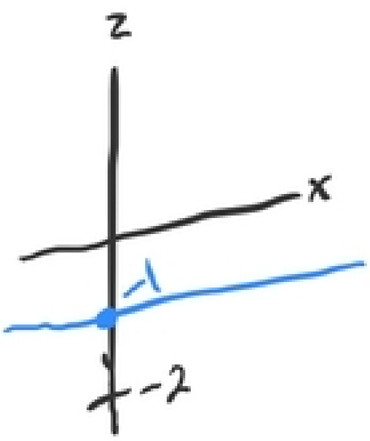
$$S_2 = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

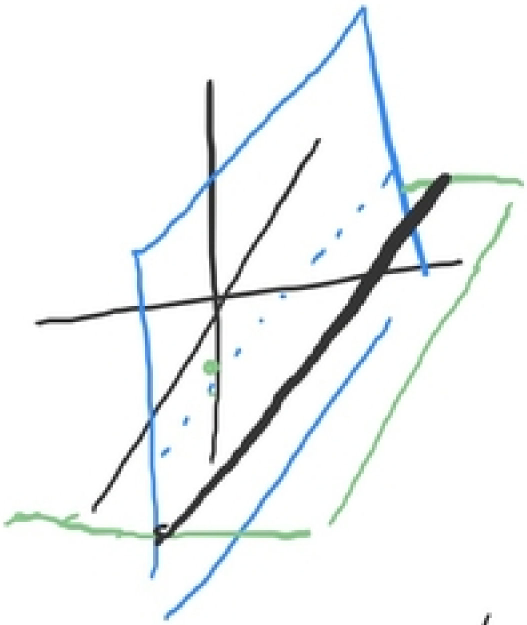
$$S_3 = \begin{pmatrix}2\\0\\1\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$R(x) = \begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} x + \begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$S(x) = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix} x + \begin{pmatrix}0\\0\\-2\end{pmatrix}$$







$$R \circ S(x) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \right) \right) + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

c) Siden R og S er speilinger er de sin egen invers

$$R \circ S \circ S \circ R(x) = R \circ Id \circ R(x)$$

$$= R \circ R(x) = Id(x) = x$$

5) 

$G = \langle \rho, \mu \mid \rho^3 = \mu^2 = Id, \underline{\mu\rho = \rho^2\mu} \rangle$

ρ, ρ^2, Id $\quad G = \{Id, \rho, \rho^2, \mu, \rho\mu, \rho^2\mu\}$

$\mu, \rho\mu, \rho^2\mu$

$\mu\rho$

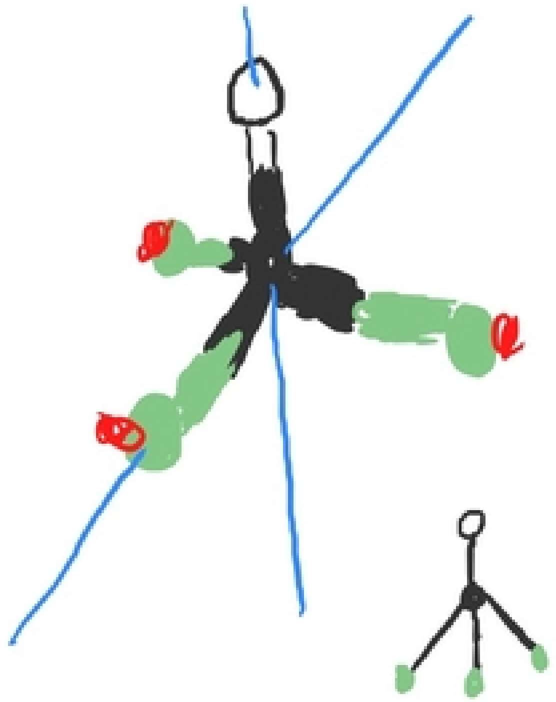
$|G| = 6$

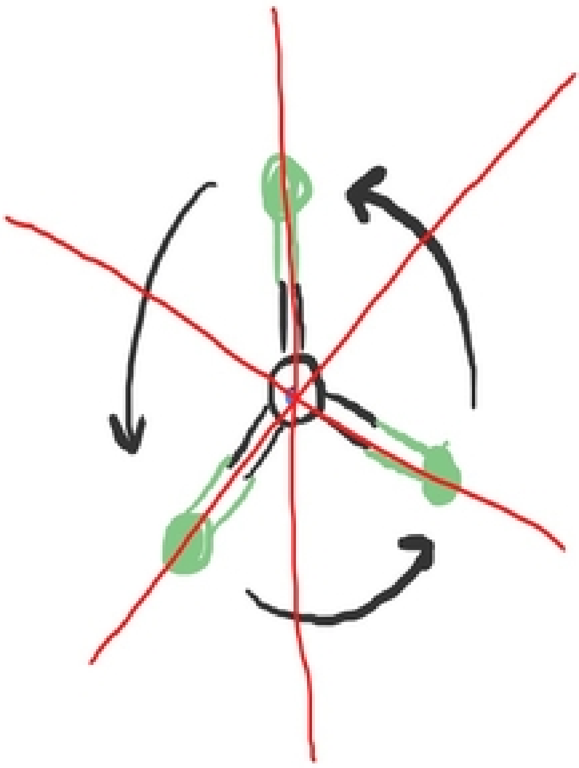
6) $|\{Id, \mu\}| = 2$ $\quad |\{\rho\mu, Id\}| = 2$

$|\{Id, \rho, \rho^2\}| = 3$ $\quad |\{\rho^2\mu, Id\}| = 2$

$|G| = 6$ $\quad |\{Id\}| = 1$

6)





$Id, \rho_{\frac{2\pi}{3}}, \rho_{\frac{4\pi}{3}}$

4) A symmetrisk matrise

egenvektorer $\vec{v}_1, \vec{v}_2, \vec{v}_3$ basis for $\mathbb{R}^3$

$\lambda_1, \lambda_2, \lambda_3$ egenverdier forskjellige fra hverandre positive

a) $x = a_1 v_1 + a_2 v_2 + a_3 v_3$ lin. komb.

Vis at $\langle x, x \rangle \geq 0$ $\quad \langle x, y \rangle_A = x^T A y$

$\langle a_1 v_1 + a_2 v_2 + a_3 v_3 , a_1 v_1 + a_2 v_2 + a_3 v_3 \rangle$

Hint $\quad v_i^T \cdot v_j = \begin{cases} \|v_i\|^2 & i = j \\ 0 & i \neq j \end{cases}$

$$= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_i a_j \langle v_i, v_j \rangle_A \right)$$

egenvektor

$$\langle v_i , v_j \rangle_A = v_i^T \cdot A \cdot v_j = v_i^T \cdot (\lambda_j v_j)$$
$$= \lambda_j v_i^T \cdot v_j$$

$$\sum_{i=1}^{3}\left(\sum_{j=1}^{3} a_i a_j \langle v_i, v_j\rangle\right)$$

$$= \sum_{i=1}^{3} a_i^2 \langle v_i, v_i\rangle = \sum_{i=1}^{3} a_i^2 \lambda_i \|v_i\|^2 .$$

$$= a_1^2 \lambda_1 \|v_1\|^2 + a_2^2 \lambda_2 \|v_2\|^2 + a_3^2 \lambda_3 \|v_3\|^2 \geq 0$$

$$\langle x, x\rangle = 0 \iff x = \vec{0}$$

Hvis $x = \vec{0} \Rightarrow a_1 = a_2 = a_3 = 0$ (fordi v_1, v_2, v_3 er lin. uavh.) Da ser vi at $\langle x, x\rangle = 0$

Hvis $\langle x, x\rangle = 0$ da må $\sum_{i=1}^{3} a_i^2 \lambda_i \|v_i\|^2$

vi vet at $\lambda_i > 0$ og $v_i \neq \vec{0}$ dermed er $\|v_i\|^2$ positiv eneste mulighet $a_1 = a_2 = a_3 = 0$

$\Rightarrow x = \vec{0}$

b) Def 6.1.1

har vi vist i oppg a) at

$\langle -,- \rangle$ er positiv definit

Må vise at $\langle -,- \rangle$ er lineær

og symmetrisk

i) $\langle au+bv, w \rangle$

$= (au+bv)^T \cdot A \cdot w$

$= (au^T + bv^T) \cdot A \cdot w$

$= a \cdot u^T \cdot A \cdot w + b \cdot v^T \cdot A \cdot w$

$= a\langle u,w \rangle + b\langle v,w \rangle$

A symmetrisk

ii) $\langle u,v \rangle = u^T \cdot A \cdot v$

$= u^T \cdot A^T \cdot v = (A \cdot u)^T \cdot v = ((A \cdot u)^T \cdot v)^T$

$= v^T \cdot ((A \cdot u)^T)^T = v^T \cdot A \cdot u = \langle v,u \rangle$