Prop: La A være en ortogonal 2×2 matrise. Da fins et reelt tall θ slik at

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{hvis } \det(A) = 1$$

og

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \quad \text{hvis } \det(A) = -1.$$

Altså representerer A enten

en rotasjon R_θ eller en

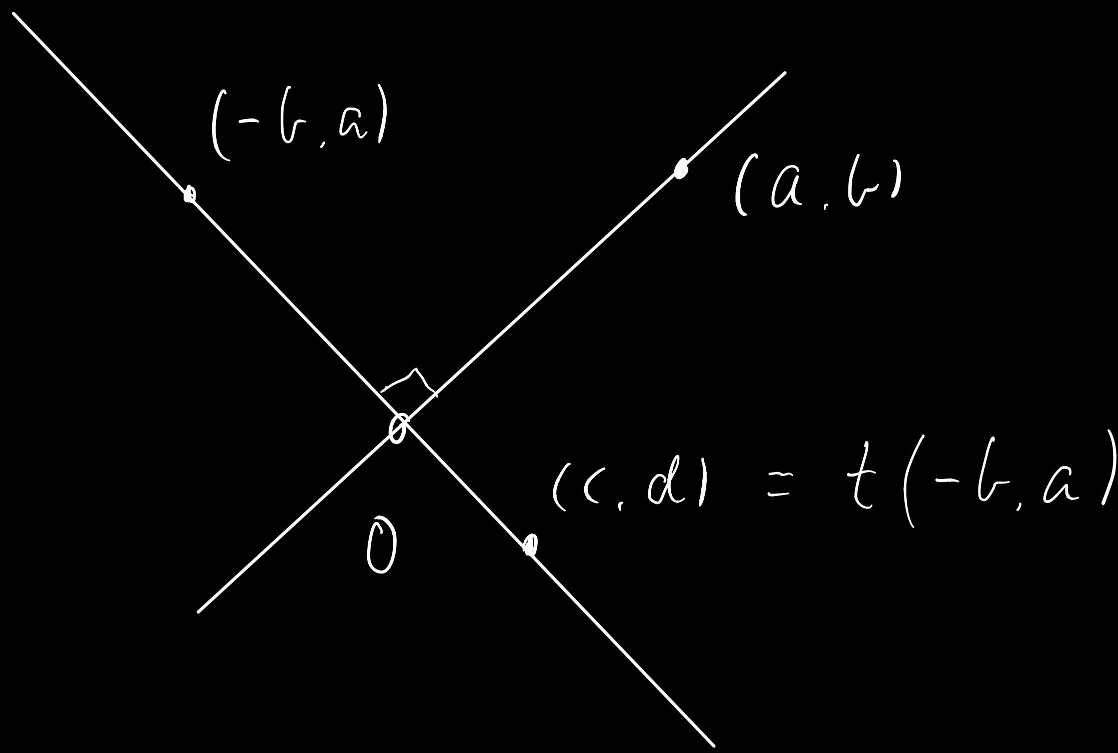
speiling S_θ.

<u>Bevis</u>: La $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Da er

$$a^2 + b^2 = 1 = c^2 + d^2$$

$$ac + bd = 0.$$



Den siste likningen gir

$$(c,d) = t(-b,a)$$

for et reelt tall t.

$$1 = c^2 + d^2 = t^2(b^2+a^2) = t^2$$

$$\Rightarrow t = \pm 1.$$

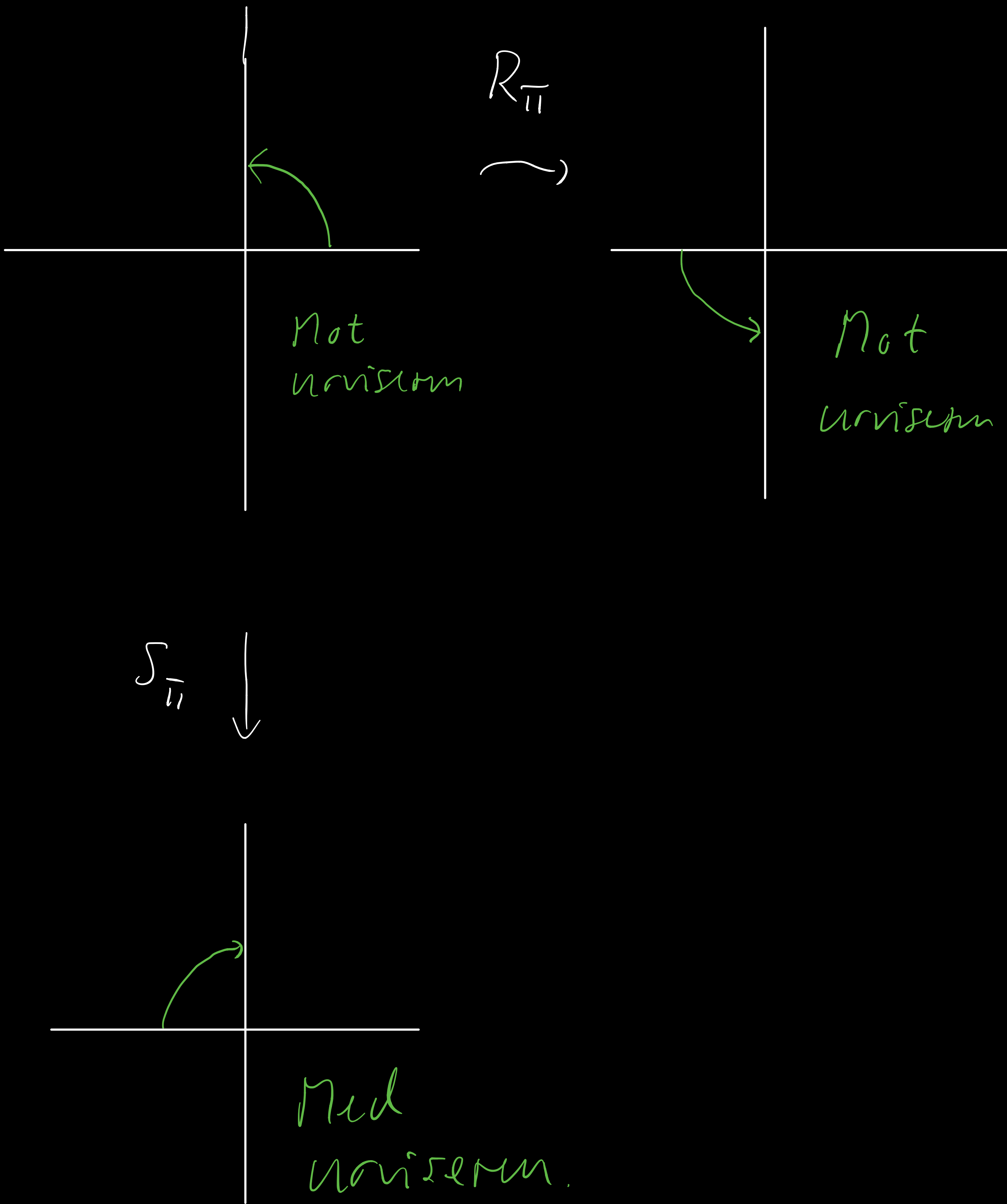
altså $(c, d) = \pm(-b, a)$.

Velg $\theta \in \mathbb{R}$ slik at

$$a = \cos\theta, \quad b = \sin\theta. \quad //$$

<u>Orienteringsbevarende</u> /

<u>orienteringsreverserende</u>



R_π
S_π
Mot urviseren
Mot urviseren
Med urviseren.

<u>Eks</u>: La $R: \mathbb{R}^2 \to \mathbb{R}^2$ være

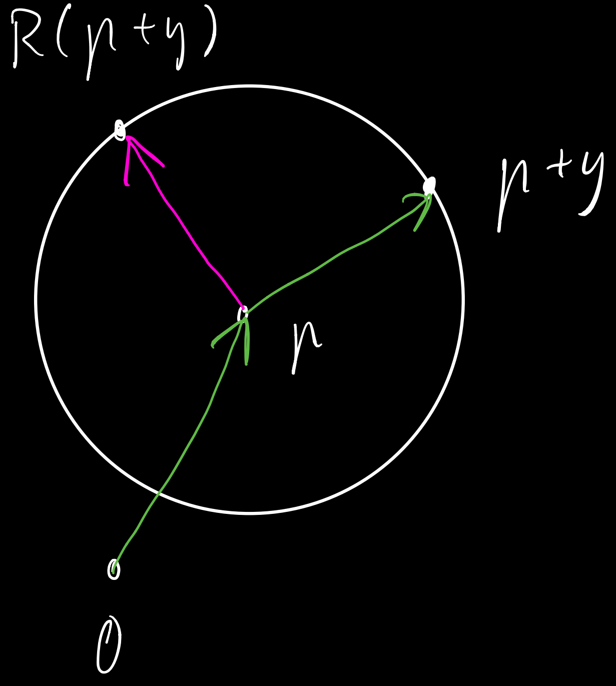
rotasjon med vinkel $\theta = \frac{\pi}{2}$

om punktet $p = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Vi ønsker å uttrykke R

på formen

$$Rx = Ax + b.$$



For alle $y \in \mathbb{R}^2$ skal

$$R(p+y) = p + R_{\frac{\pi}{2}}(y)$$

La $x = p + y$:

$$R(x) = p + R_{\frac{\pi}{2}}(x-p)$$

$$= R_{\frac{\pi}{2}}(x) + p - R_{\frac{\pi}{2}}(p)$$

$$A = R_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow R\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Householder-transformasjoner og speilinger.

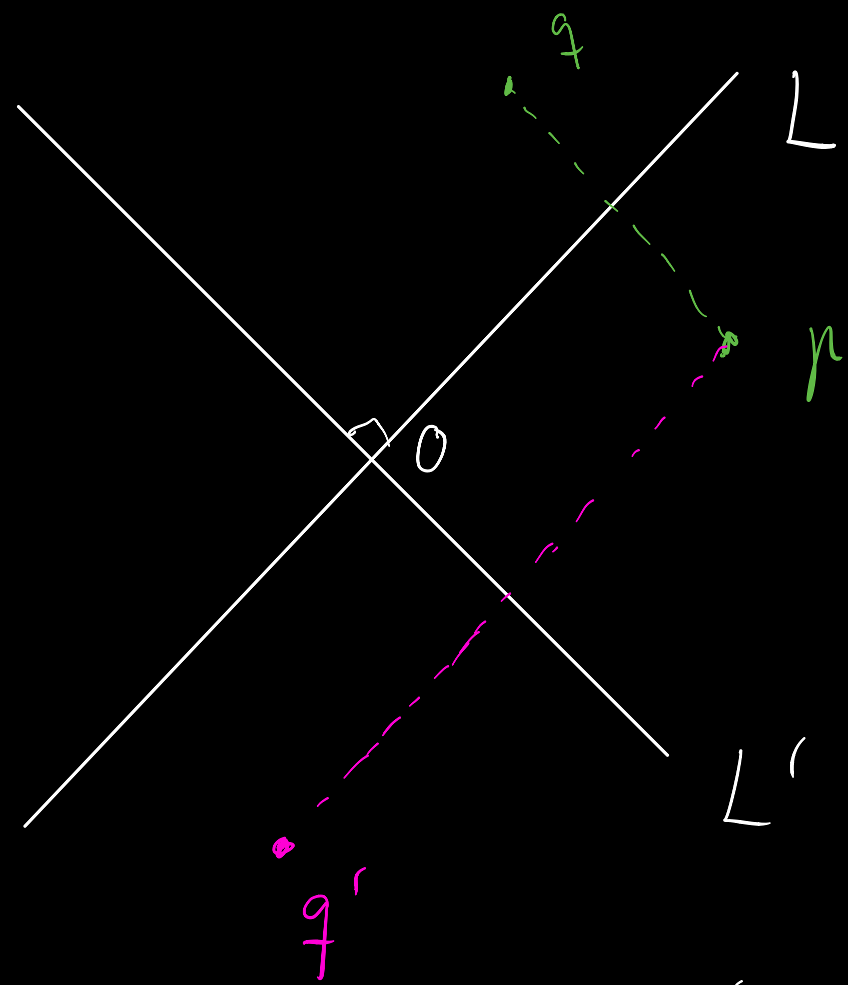
Husk: For en vektor $v \in \mathbb{R}^n$, $v \neq 0$, har vi definert Householdertransf.

$$Q_v = I - \frac{2}{\|v\|^2} v v^T$$

Har vist: For $w \in \mathbb{R}$ er

- $Q_v(x)$ = speilingen av x

i linjen gjennom 0 og v.



$q' = -q$

La nå $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\|v\| = 1$.

$$-Q_v = 2vv^T - I = \begin{pmatrix} 2v_1^2 - 1 & 2v_1v_2 \\ 2v_1v_2 & 2v_2^2 - 1 \end{pmatrix}$$

Hvis $v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, blir

$$2v_1^2 - 1 = 2\cos^2\theta - 1 = \cos 2\theta$$

$$2v_1v_2 = 2\cos\theta \cdot \sin\theta = \sin 2\theta$$

$$2v_2^2 - 1 = 2\sin^2\theta - 1 = -\cos 2\theta$$

$$\Rightarrow \; -Q_v = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$=$ matrisen til $S_{2\theta}$

<u>Eks</u>: La $m: \mathbb{R}^2 \to \mathbb{R}^2$ være speiling i en vilkårlig linje L.

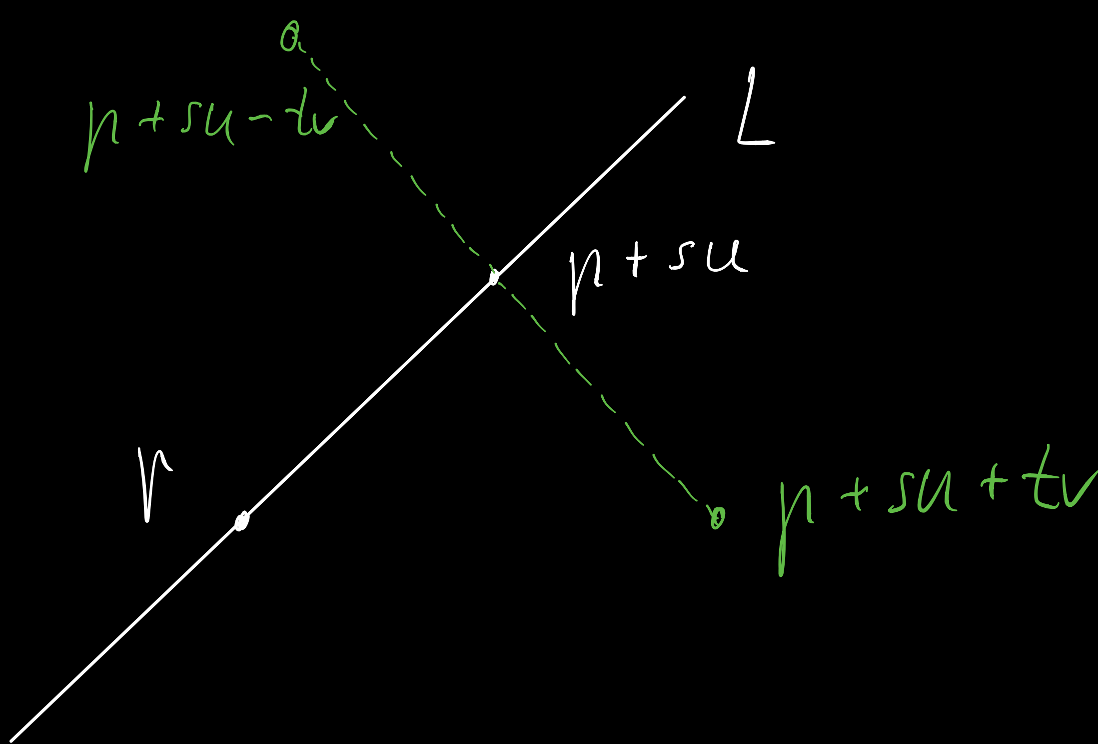
La

$$L = \{ p + su \mid s \in \mathbb{R} \},$$

hvor $p, u \in \mathbb{R}^2$, $u \neq 0$.

Velg en normalvektor v

til u: $\quad u \cdot v = 0, \quad v \neq 0.$



Da er

$$m(p+su+tv) = p+su-tv$$

$$= p + Q_v(su+tv)$$

La $x = p + su + tv$:

$$m(x) = p + Q_v(x - p)$$

$$= Q_v(x) + (I - Q_v)(p)$$

<u>Eks</u>: m = speiling i linjen

$$L = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \; 2x_1 - x_2 = 1 \right\}$$

Kan velge $v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$p = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$I - Q_v = \frac{2}{\|v\|^2} v v^T$$

$$= \frac{2}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix}$$

$$Q_v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$(I - Q_v) \cdot p = \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

Tilsammen gir dette

$$m\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$