5.1 Vektorer i $\mathbb{R}^3$.

Standardbasis for $\mathbb{R}^3$:

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Eks: $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 2i + 3j - k.$

Kryssprodukt (= vektorprodukt):

For $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

definerer vi $v \times w \in \mathbb{R}^3$ ved:

$$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\, i$$

$$- (v_1 w_3 - v_3 w_1)\, j$$

$$+ (v_1 w_2 - v_2 w_1)\, k.$$

<u>Eks</u>: $i \times j = k, \quad j \times k = i, \quad k \times i = j$

$$\begin{tikzcd} & i \arrow[dl, bend right] & \\ j \arrow[rr] & & k \arrow[ul, bend right] \end{tikzcd}$$

syklisk permutasjon.

i alminnelighet

<u>Advarsel</u>: $(u \times v) \times w \neq u \times (v \times w)$

Eks: $(i \times j) \times j = k \times j = -i$

$i \times (j \times j) = i \times 0 = 0$

<u>Prop</u>: For alle $u, v, w \in \mathbb{R}^3$, $t \in \mathbb{R}$ gjelder:

(i) $(u+v) \times w = u \times w + v \times w$

$u \times (v+w) = u \times v + u \times w$

$(tu) \times v = t \cdot (u \times v) = u \times (tv)$

(ii) $u \times v = -v \times u$

$u \times u = 0$

(iii) $u \cdot (u \times v) = 0 = v \cdot (u \times v)$

(iv) (Jakobi-identiten)

$0 = u \times (v \times w) + v \times (w \times u) + w \times (u \times v)$

(v) $\|u\|^2 \|v\|^2 = (u \cdot v)^2 + \|u \times v\|^2$

<u>Husk</u>: Hvis $u, v \neq 0$, er

$\theta = \angle(u, v)$, $0 \leq \theta \leq \pi$

definert ved

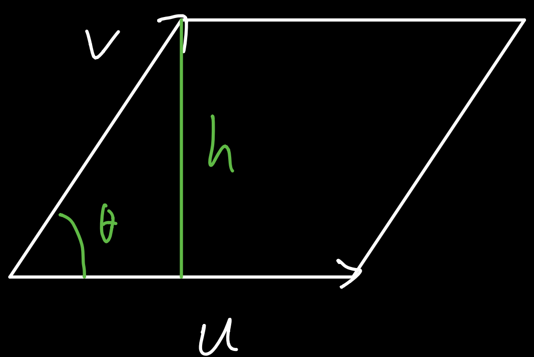
$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}.$$

$\Longrightarrow$

$$\|u \times v\|^2 = \|u\|^2 \cdot \|v\|^2 \cdot (1 - \cos^2\theta)$$

$$= \sin^2\theta \cdot \|u\|^2 \cdot \|v\|^2$$

$$\underset{\sin\theta \geq 0}{\Longrightarrow} \quad \|u \times v\| = \sin\theta \cdot \|u\| \cdot \|v\|.$$



$$h = \sin\theta \cdot \|v\|$$

$$\text{Area} = h \cdot \|u\|$$

$$= \sin\theta \cdot \|u\| \cdot \|v\|$$

$$= \|u \times v\|.$$

Def: For $u, v, w \in \mathbb{R}^3$ definerer vi <u>trippelproduktet</u>

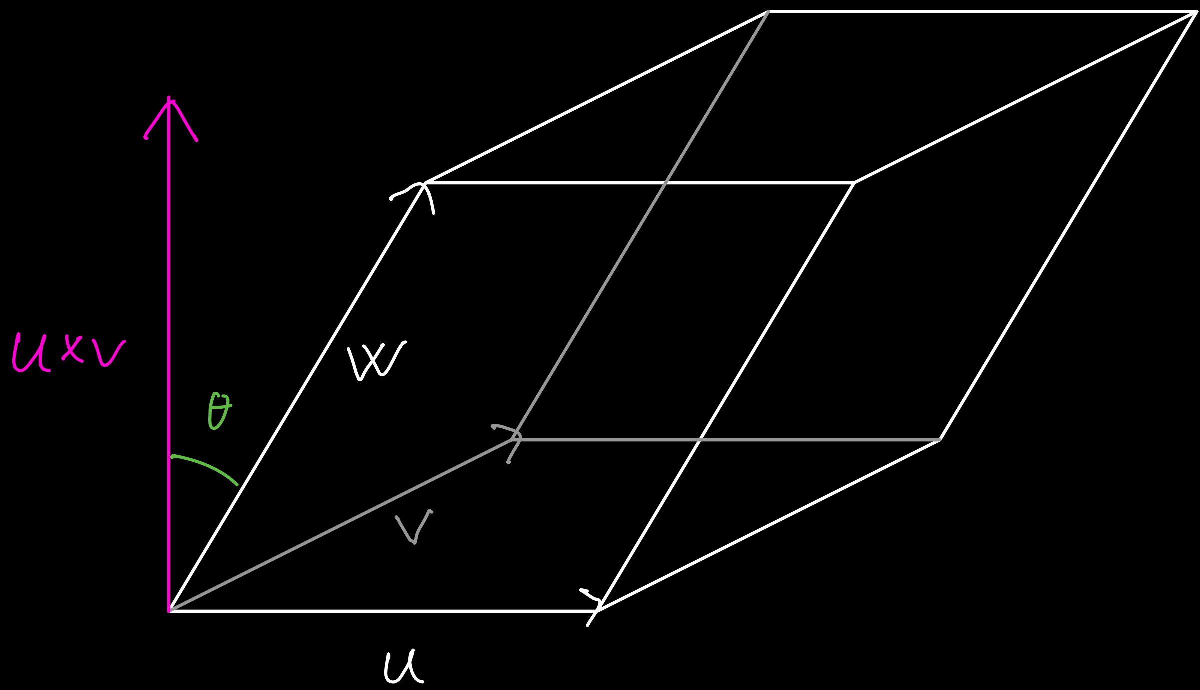
$$[u,v,w] = u \cdot (v \times w)$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Prop: $u \cdot (v \times w) = (u \times v) \cdot w$ //

Prop: $|u \cdot (v \times w)|$ er volumet av parallellepipedet utspent av u, v, w.

Bevis:



$$\text{Volum} = h \cdot A$$

$$= |\cos\theta| \cdot \|w\| \cdot \|u \times v\|$$

$$= |(u \times v) \cdot w|$$

$$= |u \cdot (v \times w)| .$$

//

Prop Hvis A er en 3×3 matrise, er

$$[Au, Av, Aw] = \det(A) \cdot [u, v, w]$$

Bevis: La B være matrisen med søyler u, v, w. Da er AB matrisen med søyler Au, Av, Aw.

$$\Rightarrow [Au, Av, Aw] = \det(AB)$$
$$= \det(A) \cdot \det(B)$$
$$= \det(A) \cdot [u, v, w]. \; //$$

Prop: Hvis A er en ortogonal 3×3 matrise og $v, w \in \mathbb{R}^3$, så er

$$(Av)\times(Aw) = \underbrace{\det(A)}_{\pm 1}\cdot(v\times w)$$

Bevis: Det er nok å vise at for alle $x \in \mathbb{R}^3$ er

$$x\cdot(Av \times Aw) = \det(A)\left(x \cdot A(v\times w)\right)$$

La $u = A^{-1}x$.

$$x \cdot (Av \times Aw) = Au \cdot (Av \times Aw)$$

$$= \det(A) \left(u \cdot (v \times w) \right)$$

$$= \det(A) \left(\underbrace{Au}_{x} \cdot A(v \times w) \right)$$

$//$