7.4 9, 10 7.6 5, 7, 15

9. $f:[a,b]\to\mathbb{R}$ kont. injektiv

Vis at f er strengt monoton

Anta at $f(a) < f(b)$ (se på $f(a) > f(b)$ etter på)

Vil vise at f er strengt voksende,

det vil si: hvis $a \leq c < d \leq b$ så er

$$f(c) < f(d).$$

–

La A = minimalverdien til f

B = maksimalverdien til f $\Rightarrow V_f = [A, B]$

Anta $f(a) = A$ og at $c < d$ men

$$f(c) \geq f(d)$$

Da finns det en $e \in [a, c]$ slik at

$f(e) = f(d)$ (SS) siden $f(a) \leq f(d) \leq f(c)$

men $e \leq c < d$ så dette

motsier at f er injektiv.

Anta så at $f(a) > A$. Da finns det en $c > a$

med $f(c) = A$, og videre finns det en $c < d \leq b$

slik at $f(d) = f(a)$ (SS) $f(c) \leq f(a) < f(b)$

Igjen er dette en motsigelse til

injektiviteten.

Så f er monotont voksende.

Når $f(a) > f(b)$ vises monotont

avtagende på samme måte

$\underline{10}$ $\quad f(x) = x\, e^{\frac{1-x^2}{2}} \qquad x \in [-1, 1]$

$$f'(x) = e^{\frac{1-x^2}{2}} + x\, e^{\frac{1-x^2}{2}}(-x) = (1-x^2)\, e^{\frac{1-x^2}{2}}$$

$(1-x^2) \geq 0 \;, \quad e^{\frac{1-x^2}{2}} > 0 \quad$ når $x \in [-1, 1]$

så $f'(x) \geq 0$ i $[-1, 1]$ og f er injektiv.

$f(-1) = -1 \cdot e^{\frac{1-1}{2}} = -1 \qquad f(1) = 1 \quad$ så $V_f = [-1, 1]$

$$\lim_{y \to 1^-} (1-y)\left[g'(y)\right]^2 \qquad\qquad y = f(x) \iff g(y) = x$$

$$= \lim_{x \to 1} (1 - f(x))\left[\frac{1}{f'(x)}\right]^2 = \lim_{x \to 1} \left(1 - x\, e^{\frac{1-x^2}{2}}\right) \frac{1}{(1-x^2)^2 e^{1-x^2}}$$

$$= \lim_{x \to 1} \frac{1 - x\, e^{\frac{1-x^2}{2}}}{(1-x^2)^2\, e^{1-x^2}} \underset{\text{l'H}}{=} \lim_{x \to 1} \frac{-(1-x^2)\, e^{\frac{1-x^2}{2}}}{\left[2(1-x^2)(-2x) + (1-x^2)^2(-2x)\right] e^{1-x^2}}$$

$$= \lim_{x \to 1} \frac{-e^{\frac{1-x^2}{2}}}{\left(2(-2x) + (1-x^2)(-2x)\right) e^{1-x^2}} = \frac{-1}{-4} = \underline{\underline{\frac{1}{4}}}$$

7.6 5

$f(x) = x \arctan x$

$f'(x) = \arctan x + \frac{x}{1+x^2}$

$f''(x) = \frac{1}{1+x^2} + \frac{1+x^2 - x(2x)}{(1+x^2)^2} = \frac{1}{1+x^2} + \frac{1-x^2}{(1+x^2)^2}$

$= \frac{1+x^2+1-x^2}{(1+x^2)^2} = \frac{2}{(1+x^2)^2} > 0$

f er konveks overalt.

$f'(x) = 0$ når $x = 0$ $\qquad \arctan x = 0$ når $x = 0$

< 0 når $x < 0$ avtagende $\qquad < 0 \quad x < 0$

> 0 når $x > 0$ voksende $\qquad > 0 \quad x > 0$

$\frac{x}{1+x^2} = 0$ når $x = 0$

$< 0 \quad x < 0$

$> 0 \quad x > 0$

f er definert overalt så den har ikke vertikale asymptoter.

$f(x) = x \arctan x \longrightarrow \pm\infty$ når $x \to \pm\infty$

så f har heller ikke horisontal asymptote.

$f'(x) = \arctan x + \frac{x}{1+x^2} \longrightarrow \pm\frac{\pi}{2}$ når $x \to \pm\infty$

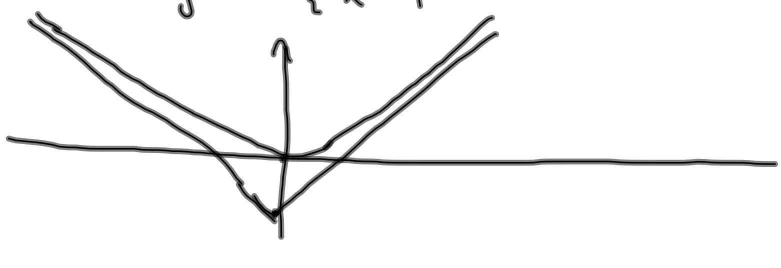
$\lim_{x\to\infty} \underline{f(x) - \frac{\pi}{2}x} = \lim_{x\to\infty} x\arctan x - \frac{\pi}{2}x$

$= \lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{\frac{1}{x}} \overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2}{1+x^2}$

$= \lim_{x\to\infty} \frac{-1}{\frac{1}{x^2}+1} = \underline{-1}$

så $y = \frac{\pi}{2}x \underline{-1}$ er skråasymptote.

Tilsvarende er $y = -\frac{\pi}{2}x - 1$



7. $$\frac{1+x}{1+x^2} = 2\arctan x$$

$$g(x) = \frac{1+x}{1+x^2} - 2\arctan x$$

$$g'(x) = \frac{1+x^2-(1+x)(2x)}{(1+x^2)^2} - \frac{2}{1+x^2}$$

$$= \frac{1-2x-x^2-2-2x^2}{(1+x^2)^2} = \frac{-1-2x-3x^2}{(1+x^2)^2}$$

$$= \frac{-1-2x-x^2-2x^2}{(1+x^2)^2} = \frac{-(1+x)^2-2x^2}{(1+x^2)^2} < 0$$

så g er injektiv og kont.

$$g\left(\tfrac{1}{3}\sqrt{3}\right) = \frac{1+\frac{1}{\sqrt{3}}}{1+\frac{1}{3}} - 2\cdot\frac{\pi}{6} = \frac{3}{4}\left(1+\frac{1}{\sqrt{3}}\right) - \frac{\pi}{3} > 0$$

$$g(1) = 1 - 2\arctan 1 = 1 - 2\cdot\frac{\pi}{4} = 1-\frac{\pi}{2} < 0$$

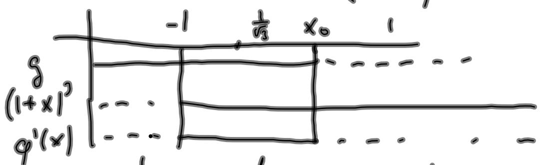
av SS finis det en $x_0 \in \left(\frac{1}{\sqrt{3}}, 1\right)$

slik at $g(x_0) = 0$.

7b $$\varphi(x) = \frac{\arctan x}{(1+x)^2}$$

$$\varphi'(x) = \frac{\frac{1}{1+x^2}(1+x)^2 - \arctan x\cdot 2(1+x)}{(1+x)^4}$$

$$= \frac{\frac{1+x}{1+x^2} - 2\arctan x}{(1+x)^3} \leftarrow g(x)$$



avt. voks avt.

$$\varphi(x) = \frac{\arctan x}{(1+x)^2} \quad \begin{matrix} < 0 & \text{når } x < 0 \\ > 0 & \text{når } x > 0 \end{matrix}$$

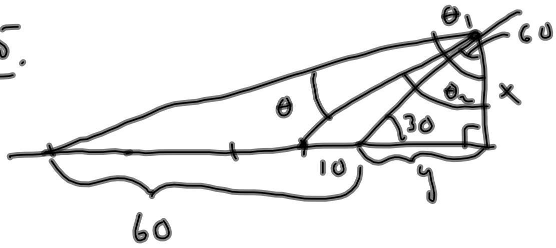
så φ har maks verdi når $x > 0$.

så $$\varphi_{max} = \varphi(x_0) = \frac{\arctan x_0}{(1+x_0)^2}$$

$$2\arctan x_0 = \frac{1+x_0}{1+x_0^2}$$

$$= \frac{\frac{1}{2}\frac{1+x_0}{1+x_0^2}}{(1+x_0)^2} = \frac{1}{2(1+x_0)(1+x_0^2)}$$

15.



$$\theta = \theta_1 - \theta_2$$

$$\frac{y}{x} = \tan 60^\circ \Rightarrow y = x\sqrt{3}$$

$$\tan\theta_1 = \frac{60+y}{x} = \frac{60}{x} + \sqrt{3}$$

$$\tan\theta_2 = \frac{10+y}{x} = \frac{10}{x} + \sqrt{3}$$

$$v(x) = \theta = \theta_1 - \theta_2 = \arctan\left(\frac{60}{x} + \sqrt{3}\right) - \arctan\left(\frac{10}{x} + \sqrt{3}\right)$$

$$v'(x) = \frac{1 \cdot -\frac{60}{x^2}}{1 + \left(\frac{60}{x} + \sqrt{3}\right)^2} - \frac{-\frac{10}{x^2}}{1 + \left(\frac{10}{x} + \sqrt{3}\right)^2}$$

$$= \frac{-60}{x^2 + 60^2 + 120\sqrt{3}\,x + 3x^2} + \frac{10}{x^2 + 10^2 + 20\sqrt{3}\,x + 3x^2}$$

$v'(x) = 0$

$$\Rightarrow \quad \frac{10}{4x^2 + 10^2 + 20\sqrt{3}\,x} = \frac{60}{4x^2 + 60^2 + 120\sqrt{3}\,x}$$

$$\rightsquigarrow \quad x = \sqrt{150}$$

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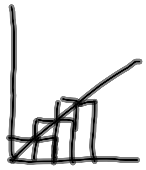
$\underline{8.2}$ 5 $\quad f : [0,1] \to \mathbb{R} \qquad f(x) = x$

$\Pi_n = [0, \frac{1}{n}, \ldots \frac{n-1}{n}, \frac{n}{n}]$



a) $\Phi(\Pi_n) = f(\frac{1}{n})\cdot\frac{1}{n} + f(\frac{2}{n})\cdot\frac{1}{n} + \quad + f(\frac{n}{n})\cdot\frac{1}{n}$

$= \frac{1}{n^2}(1 + 2 + 3 + \quad + n)$

$= \frac{1}{n^2}\frac{(n+1)\cdot n}{2} = \underline{\frac{n+1}{2n}}$

$N(\Pi_n) = f(0)\cdot\frac{1}{n} + f(\frac{1}{n})\cdot\frac{1}{n} + \quad + f(\frac{n-1}{n})\cdot\frac{1}{n}$

$= \frac{1}{n^2}(0 + 1 + 2 + \quad + (n-1))$

$= \frac{1}{n^2}\frac{(n-1)\cdot n}{2} = \underline{\frac{n-1}{2n}}$

$\lim_{n\to\infty} \Phi(\Pi_n) = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$

$\lim_{n\to\infty} N(\Pi_n) = \lim_{n\to\infty}\frac{n-1}{2n} = \lim_{n\to\infty}\frac{1-\frac{1}{n}}{2} = \frac{1}{2}$

$\overset{\frac{1}{2}}{\overset{\shortparallel}{\lim_{n\to\infty} N(\Pi_n)}} \leq \overline{\int_0^1} x\,dx \leq \lim_{n\to\infty}\Phi(\Pi_n) = \frac{1}{2} \Rightarrow \overline{\int} = \frac{1}{2}$

tilsvarende er $\underline{\int} = \frac{1}{2}$ og derfor også $\underline{\int = \frac{1}{2}}$

8.3

3 d

$$\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2} = \int_0^1 \frac{\frac{1}{2}\,du}{1+u^2}$$

$u = 2x$

$du = 2\,dx \qquad dx = \frac{1}{2}\,du$

$$= \frac{1}{2}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\Big[\arctan u\Big]_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \underline{\frac{\pi}{8}}$$

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5 a) $f(x) = \int_0^x e^{-t^2}\,dt$

$f'(x) = e^{-x^2}$

b) $\frac{d}{dx}\int_1^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$

c) - -

6 a) $G(x) = \int_a^{g(x)} f(t)\,dt$

$\frac{d}{dx} G(x) = \frac{d}{dx}\left(G(\underset{u}{\overset{\shortparallel}{g(x)}})\right) = \frac{d}{du}G(u)\cdot\frac{d}{dx}g(x)$

$= f(g(x))\cdot g'(x)$

b) i) $\frac{d}{dx}\int_0^{\sin x} t e^{-t}\,dt = \sin x\, e^{-\sin x}\cdot\cos x$

$= \sin x\cos x\, e^{-\sin x}$

ii) $\frac{d}{dx}\int_0^{\sqrt{x}} e^{-t^2}\,dt = e^{-(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} = \underline{\frac{e^{-x}}{2\sqrt{x}}}$

iii) $\frac{d}{dx}\int_{\sin x}^0 \frac{dt}{\sqrt{1-t^2}} = \frac{d}{dx}\left(-\int_0^{\sin x}\frac{dt}{\sqrt{1-t^2}}\right)$

$= -\frac{1}{\sqrt{1-\sin^2 x}}\cdot\cos x = -\frac{\cos x}{\cos x} = \underline{-1}$

$-\int_0^{\sin x}\frac{dt}{\sqrt{1-t^2}} = -\left[\arcsin t\right]_0^{\sin x} = \underline{-x}$

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