Plenum 2/11

8.5: 1a, 2, 4, 5

8.6: 1a, e, f, 3, 5c, 7b, c, e, 9, 11 a, c, 15

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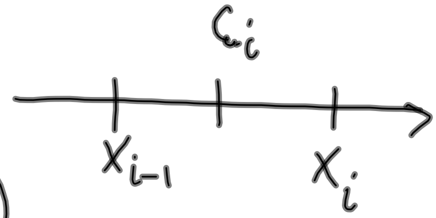
8.5: 2.) $f(x) = x$, $\Pi_n = \{0, x_1, \dots, x_n\}$, $x_n = a$. U_n s.a. $c_i = \dfrac{x_i + x_{i-1}}{2}$

Finn $R(\Pi_n, U_n)$:



$$R(\Pi_n, U_n) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} c_i (x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{(x_i + x_{i-1})}{2}(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \frac{x_i^2 - x_{i-1}^2}{2} = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)$$

$$\cancel{x_1^2} - x_0^2 + \cancel{x_2^2} - \cancel{x_1^2} + \cancel{x_3^2} - \cancel{x_2^2} + \dots \cancel{-\dots} + x_n^2 - \cancel{x_{n-1}^2}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_{i-1}^2\right)$$

$$= \frac{1}{2}\left(\cancel{\sum_{i=1}^{n-1} x_i^2} + x_n^2 - x_0^2 - \cancel{\sum_{i=1}^{n-1} x_i^2}\right)$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \underline{\underline{\frac{a^2}{2}}}$$

Det viser seg at så lenge U_n velges på denne måten, så har partisjonen Π_n ingenting å si. Uansett er $R(\Pi_n, U_n) = \frac{a^2}{2}$.

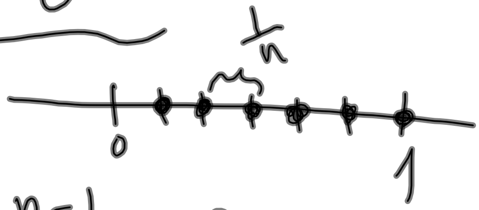
Fra Kor. 8.5.4, er

$$\int_0^a x\,dx = \lim_{n \to \infty} R(\Pi_n, U_n) = \frac{a^2}{2}$$

(fordi man kan velge en følge av partisjoner Π_n s.a. $|\Pi_n| \longrightarrow 0$ der utvalget er som over.)

4.) $\underbrace{\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}}\left(\sum_{i=1}^{n}\sqrt{i}\right) = \frac{2}{3}}$:

Se på: $\int_0^1 \sqrt{x}\,dx$.



La $\Pi_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$ være en partisjon av $[0,1]$, og la $U_n = \left\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$:

$$R(\Pi_n, U_n) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \sqrt{c_i}(x_i - x_{i-1}) = \sum_{i=1}^{n} \sqrt{\frac{i}{n}}\left(\frac{i}{n} - \frac{i-1}{n}\right)$$

$$= \sum_{i=1}^{n} \sqrt{i}\,\frac{1}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n}\sqrt{i}$$

Derfor er $\frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n}\sqrt{i}$ en Riemannsum for $\int_0^1 \sqrt{x}\,dx$. Merk: Når $n\to\infty$, vil $|\Pi_n| \to 0$.

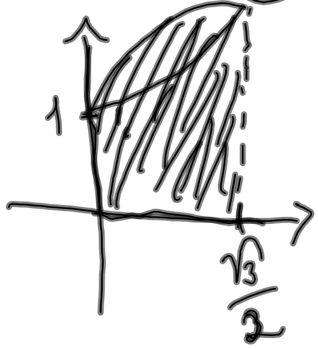
Fra Korollar 8.5.4, vil

$$\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n}\sqrt{i} = \int_0^1 \sqrt{x}\,dx = \int_0^1 x^{\frac{1}{2}}\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_{x=0}^{1} = \underline{\underline{\frac{2}{3}}}$$

8.6: 1) e) $y = \frac{1}{\sqrt{1-x^2}}$, x-aksen, y-aksen,

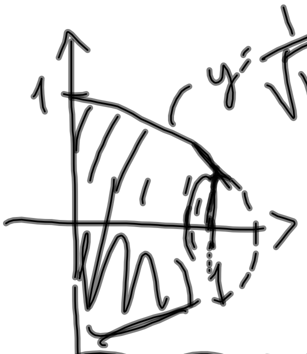
$x = \frac{\sqrt{3}}{2}$:



$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\arcsin x\right]_{x=0}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi}{3} - 0 = \underline{\underline{\frac{\pi}{3}}}$$

5.) c) $f(x)=\frac{1}{\sqrt{1+x^2}}, \; x=0, \; x=1$:



$$V=\int_0^1 \pi f^2(x)\,d$$

$$=\int_0^1 \pi \frac{1}{1+x^2}\,dx = \pi\left[\arctan x\right]_{x=0}^{1}$$

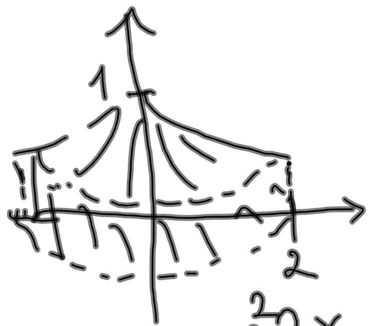
$$=\pi\left(\frac{\pi}{4}-0\right)$$

$$y^2(x) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1^2}{\left(\sqrt{1+x^2}\right)^2} = \frac{1}{1+x^2}$$

$$=\frac{\pi^2}{4}$$

7.) c) $\underbrace{y=\frac{1}{1+x^2},\ x=0,\ x=2}$

$$V=\int_0^2 2\pi x\, f(x)\,dx$$



$$=\int_0^2 2\pi x\,\frac{1}{1+x^2}\,dx$$

$$=\pi\int_0^2 \frac{2x}{1+x^2}\,dx=\pi\left[\ln(1+x^2)\right]_{x=0}^{2}$$

$$=\pi\left(\ln(1+4)-\underbrace{\ln(1+0)}_{0}\right)=\underline{\underline{\pi\ln 5}}$$

11.) a) $y = \frac{x^2}{2} - \frac{1}{4}\ln x, \; x=1, \; x=e:$

$$L = \int_1^e \sqrt{1+[f'(x)]^2}\,dx$$

Sek. 8.6.7

$$= \int_1^e \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx$$

$$= \int_1^e \sqrt{1+x^2-\frac{2\cancel{x}}{4\cancel{x}}+\frac{1}{16x^2}}\,dx$$

$$= \int_1^e \sqrt{\frac{1}{2}+x^2+\frac{1}{16x^2}}\,dx = \int_1^e \sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx$$

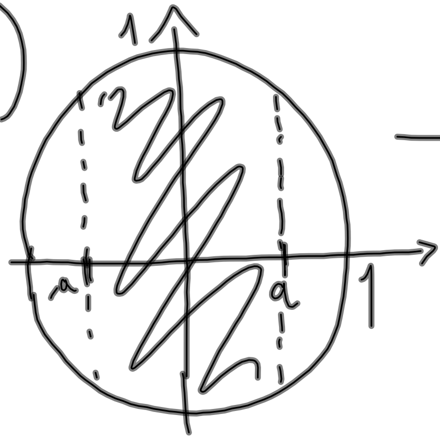
$$= \int_1^e \left(x+\frac{1}{4x}\right)dx = \left[\frac{1}{2}x^2+\frac{1}{4}\ln 4x\right]_{x=1}^{e}$$

$$= \frac{1}{2}e^2+\frac{1}{4}\ln(4e)-\frac{1}{2}-\frac{1}{4}\ln(4)$$

$$= \frac{1}{2}e^2+\frac{1}{4}(\cancel{\ln(4)}+\underbrace{\ln e}_{1})-\frac{1}{2}-\cancel{\frac{1}{4}\ln 4}$$

$$= \frac{1}{2}e^2-\frac{1}{4}$$

15.)



siden over og under x-aksen

$$V_{\text{omdreining}} = 2\int_a^1 2\pi x \underbrace{\sqrt{1-x^2}}\,dx$$

Kule m/ radius 1 og sentrum origo kan beskrives v/

$y^2 + x^2 = 1$

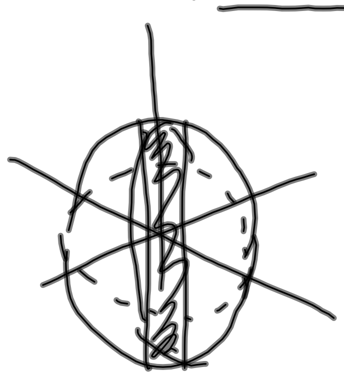
$y = \pm\sqrt{1-x^2}$

$$= 2\pi \int_a^1 2x\sqrt{1-x^2}\,dx$$

$\sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

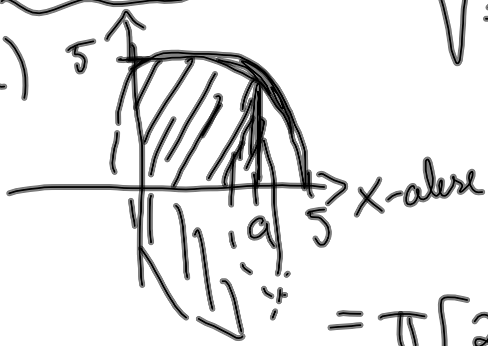
$$= 2\pi \left[-(1-x^2)^{\frac{3}{2}} \frac{2}{3} \right]_{x=a}^{1}$$

$$= \dots = \frac{4\pi}{3}(1-a^2)^{\frac{3}{2}}$$



26.) $a \in [0,5]$, $f(x) = \sqrt{25-x^2}$, x-aksen, y-aksen og f og $x = a$ dreies om x-aksen:

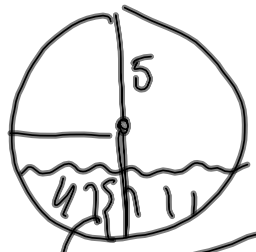
a)



$$V = \int_0^a \pi \left(\sqrt{25-x^2}\right)^2 dx = \int_0^a \pi (25 - x^2) dx = \pi \left[25x - \frac{1}{3}x^3\right]_{x=0}^{a}$$

$$= \pi\left(25a - \frac{1}{3}a^3\right) = \pi a \left(25 - \frac{1}{3}a^2\right)$$

b)



2m: 1
Tanken tømmes
m/ $\frac{1}{2}$ m^3/min

$$V(a) = \pi a\left(25 - \frac{1}{3}a^2\right)$$

(a)

Vanndybde = 2m $\Rightarrow$

$$a = 5 - 2 = 3$$

Se på a som en funksjon av tiden:

$$V(t) = V(a(t)) = \pi a(t)\left(25 - \frac{1}{3}a^2(t)\right)$$

Koblede hastigheter:

$$\cancel{V'(t) = \pi\left(25a'(t) - \frac{2}{3}a(t)a'(t)\right)}$$

$$V(t) = \pi\left(25a(t) - \frac{a^3(t)}{3}\right)$$

$$V'(t) = \pi\left(25a'(t) - \frac{\cancel{3}a^2(t)a'(t)}{\cancel{3}}\right)$$

$$= \pi a'(t)\left(25 - a^2(t)\right)$$

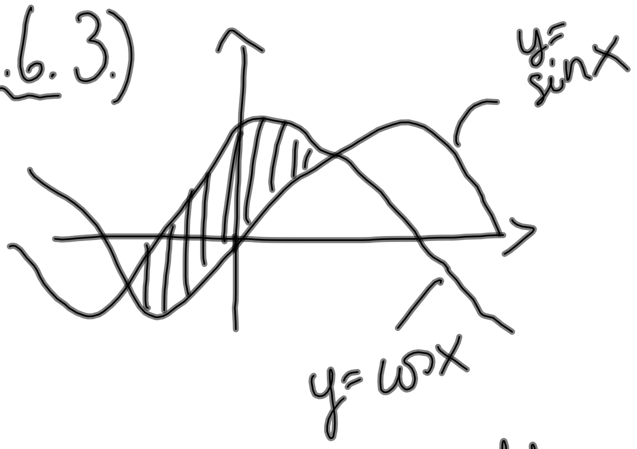
V/ tid der $a(t) = 3$, er $V'(t) = \frac{1}{2}$:

$$\frac{1}{2} = \pi a'(t)(25 - 9)$$

$$a'(t) = \frac{1}{32\pi}$$

Så vannhøyden avtar med $\frac{1}{32\pi}$ m/min på denne tiden.

8.6.3)



Må finne skjæringspkt.:

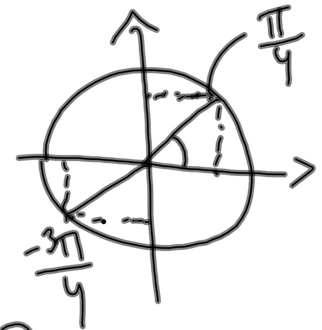
$$\sin x = \cos x \Leftrightarrow x = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$$

Ser på figuren at våre skjæringspunkter er

$x = \frac{\pi}{4}$ $(k=0)$, $x = \frac{\pi}{4} + (-1)\pi = -\frac{3\pi}{4}$ $(k=-1)$.

I intervallet $[-\frac{3\pi}{4}, \frac{\pi}{4}]$ så er $\sin x < \cos x$.

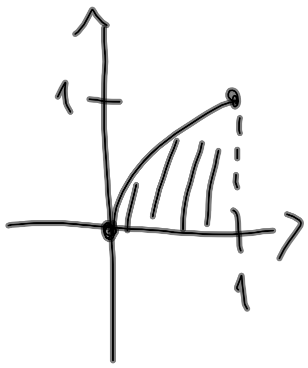
$$A = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos x\,dx - \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \sin x\,dx$$



$$= \left[\sin x\right]_{x=-\frac{3\pi}{4}}^{\frac{\pi}{4}} - \left[-\cos x\right]_{x=-\frac{3\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = \underline{\underline{2\sqrt{2}}}$$

8.6: 1.) a) $y = x^4$, x-aksen, $x=1$:

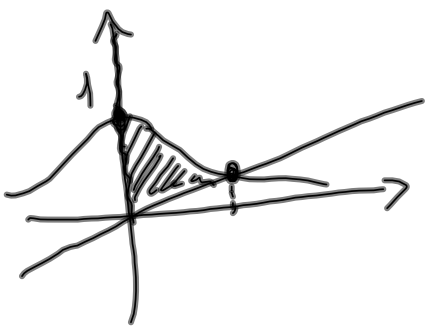


$y = x^4 = 0 \Leftrightarrow x = 0$

Areal:

$$\int_0^1 x^4\,dx = \left[\frac{1}{5}x^5\right]_{x=0}^{1} = \underline{\underline{\frac{1}{5}}}$$

f) $y = \frac{1}{1+x^2}$, $y = \frac{x}{2}$, y-aksen:



$$\frac{x}{2} = \frac{1}{1+x^2}$$

$$x(1+x^2) = 2 \quad (x + x^3 = 2)$$

Ser at $x=1$ er en løsning, grafen skjærer kun én gang, så derfor må $x=1$ funke.

$$A = \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{2}\,dx$$

(ser fra fig. at i $[0,1]$, så er $\frac{1}{1+x^2} \geq \frac{x}{2}$)

$$= [\arctan x]_{x=0}^{1} - \frac{1}{2}\left[\frac{1}{2}x^2\right]_{x=0}^{1}$$

$$= \frac{\pi}{4} - 0 - \frac{1}{4}(1-0) = \underline{\underline{\frac{\pi - 1}{4}}}$$

8.6: 7) e) $y = \sin x^2,\ x = 0,\ x = \sqrt{\pi}$:

$$V = \int_0^{\sqrt{\pi}} 2\pi x \sin x^2\, dx = \pi \int_0^{\sqrt{\pi}} 2x \sin x^2\, dx$$

$$= \pi \left[-\cos(x^2) \right]_{x=0}^{\sqrt{\pi}} = \pi\left(\cos 0 - \cos\left((\sqrt{\pi})^2\right)\right)$$

$$= \pi(1 - (-1)) = \underline{\underline{2\pi}}$$