Plenum 26/10

8.3: 1a), c), e), h), 3 a), b), c), g), 4, 7a), b), 9, 13

8.4: 1, 2, 3, 5

8.3: Analysens fundamentalteorem

1) a) $\int_0^{\pi} \sin x \, dx = [-\cos x]_{x=0}^{\pi} = -\cos\pi + \cos 0$
$= 1 + 1 = \underline{\underline{2}}$

c) $\int_0^1 e^{-x} dx = [-e^{-x}]_{x=0}^{1} = -e^{-1} + e^{-0} = \underline{\underline{1 - \frac{1}{e}}}$

e) $\int_1^e \frac{1}{x} dx = [\ln x]_{x=1}^{e} = \ln e - \ln 1 = 1 - 0 = \underline{\underline{1}}$

h) $\int_1^9 \sqrt{x}^3 dx = \int_1^9 x^{\frac{3}{2}} dx = [\frac{2}{5} x^{\frac{5}{2}}]_{x=1}^{x=9}$
$= \frac{2}{5}(\sqrt{9}^5 - \sqrt{1}^5) = \frac{2}{5}(3^5 - 1) = \underline{\underline{\frac{484}{5}}}$

$3 \cdot 3 = 9$
$3^4 = 9 \cdot 9 = 81$
$81 \cdot 3 = 240 + 3$
$= 243$

3) a) $\int_0^{\frac{2\pi}{3}} \sin(x + \frac{\pi}{3}) dx = [-\cos(x + \frac{\pi}{3})]_{x=0}^{\frac{2\pi}{3}}$

$= -\cos(\pi) + \cos(\frac{\pi}{3}) = 1 + \frac{1}{2} = \underline{\underline{\frac{3}{2}}}$

b) $\int_0^2 e^{3x+2} dx = [\frac{1}{3} e^{3x+2}]_{x=0}^{2}$
$= \frac{1}{3}(e^{6+2} - e^2) = \frac{e^8 - e^2}{3} = \underline{\underline{\frac{e^2(e^6 - 1)}{3}}}$

c) $\int_1^4 \frac{1}{2x+1} dx = [\ln(2x+1)\frac{1}{2}]_{x=1}^{4}$

$= \frac{1}{2}(\ln(9) - \ln(3)) = \frac{1}{2}(2\ln(3) - \ln 3)$
$= \underline{\underline{\frac{\ln 3}{2}}}$

g) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\frac{1}{\cos^2 x} + \frac{1}{e^{7x}}) dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-7x} dx$

$$= [\tan x]_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}} + [e^{-7x}(-\tfrac{1}{7})]_{x=-\frac{\pi}{4}}^{x=\frac{\pi}{4}}$$

$$= 1+1+(-\tfrac{1}{7})(e^{-\frac{7\pi}{4}} - e^{\frac{7\pi}{4}})$$

$$= 2 + \frac{e^{\frac{7\pi}{4}} - e^{-\frac{7\pi}{4}}}{7}$$

4) a) $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} x\cos(x^2)\,dx = [\sin(x^2)\tfrac{1}{2}]_{x=-\sqrt{\pi}}^{\sqrt{\pi}}$

$$= \tfrac{1}{2}(\sin(\pi) - \sin(\pi)) = \underline{\underline{0}}$$

b) $\int_0^1 \frac{4x}{1+x^2}\,dx = [2\ln(1+x^2)]_{x=0}^{1}$

$$= 2(\ln(1+1^2) - \ln(1+0^2))$$

$$= 2(\ln 2 - \ln 1) = \underline{\underline{2\ln 2}}$$

c) $\int_0^1 x^2 e^{x^3}\,dx = [\tfrac{1}{3}e^{x^3}]_{x=0}^{1} = \tfrac{1}{3}(e^1 - e^0) = \underline{\underline{\tfrac{1}{3}(e-1)}}$

d) $\int_0^{\pi} \cos x\, e^{\sin x}\,dx = [e^{\sin x}]_{x=0}^{\pi} = e^{\sin\pi} - e^{\sin 0}$

$$= e^0 - e^0 = \underline{\underline{0}}$$

$\cos x > 0$ for $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ → $\boxed{\text{Trenger ikke } |\cdot|}$

e) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = [-\ln[\cos x]]_{x=-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$$= -\ln(\cos\tfrac{\pi}{4}) + \ln(\cos(-\tfrac{\pi}{4})) = \underline{\underline{0}}$$

7) a) $\lim_{x \to 0} \frac{\int_0^x e^{-t^2} dt}{x} = \left[\frac{d}{dx}\left[\int_0^x e^{-t^2} dt\right]\right]_{x=0}$

Fra def. av den deriverte

$= e^{-0^2} = \underline{\underline{1}}$

Analysens fundamentalthm.

b) $\lim_{x \to \infty} \frac{\int_1^x e^{\frac{1}{t}} dt}{x^2}$

7) a) $\lim_{x \to 0} \frac{\int_0^x e^{-t^2} dt}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}\left[\int_0^x e^{-t^2} dt\right]}{1}$

$\boxed{\text{"}\frac{0}{0}\text{"} : \text{L'Hôpital}}$

$= \lim_{x \to 0} e^{-x^2} = e^0 = \underline{\underline{1}}$

Analysens fundamentalteorem

b) $\lim_{x \to \infty} \frac{\int_1^x e^{\frac{1}{t}} dt}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}\left[\int_1^x e^{\frac{1}{t}} dt\right]}{2x}$

"$\frac{\infty}{\infty}$" : L'Hôpital

$= \lim_{x \to \infty} \frac{e^{\frac{1}{x}}}{2x} = \underline{\underline{0}}$

Analysens fundamentalthm

9) Anta f kont. Vis at $\exists\, c \in (a, b)$:

$$\int_a^b f(x)dx = f(c)(b-a)$$

Middelverdisetningen for integraler

F kont. og deriverbar

Analysens fundam... th

La $F(x) := \int_d^x f(t)dt$ der $d<a$. Da er $F'(x) = f(x)$.

Merk at F er veldef. siden f er kont.

Se på intervallet $[a, b]$. Da gir middelverdisetningen at det fins $c \in (a, b)$ s. a.

$$F'(c) = \frac{F(b)-F(a)}{b-a}$$, dvs.

$$f(c)(b-a) = \int_d^b f(t)\,dt - \int_d^a f(t)\,dt$$

$$f(c)(b-a) = \int_a^b f(t)\,dt$$

(3.) g: pos, monotont voksende, kont. funksjon på $[0,\infty)$.

Def: $\quad h(x) := \int_0^x g(t)\,dt$

a) Vis at h er pos. og voksende:

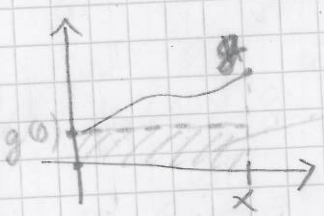
~~Fra Korollar 8.2.4 er (siden g er monoton)~~

$$\cancel{h(x) = \lim_{n\to\infty} \sum_{i=1}^{n} g(y_i)\Delta y}$$

~~der $0 = y_0 < y_1 < \dots < y_n = x$ er en inndeling av $[0,x]$ i like store deler.~~

h er positiv fordi:

$$\cancel{h(x) = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} \underbrace{g(y_i)}_{>0} \underbrace{\Delta y}_{>0}}_{>0}}$$



$$h(x) \geq \overbrace{(x-0)}^{>0}\, \overbrace{g(0)}^{\geq 0} > 0$$

den grovest mulige nedre trappesummen (er nedre trappesum pga. g monotont voksende)

Fra def. av integralet

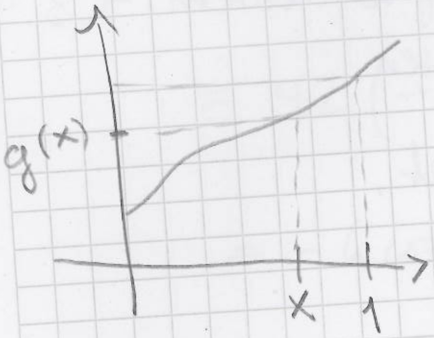
h er voksende fordi:

Anta $z > x$. Vil vise at $h(z) > h(x)$.

$$h(z) = \int_0^z g(t)\,dt = \int_0^x g(t)\,dt + \underbrace{\int_x^z g(t)\,dt}_{>0 \text{ fra samme arg. som over}}$$

$$> \int_0^x g(t)\,dt = h(x)$$

b) Vis at $h(x) \le g(x)\ \forall x \in [0,1]$.



$$h(x) = \int_0^x g(t)\,dt$$

$$\le (x-0)\,g(x)$$

$$= x\,g(x) \le g(x)$$

groveste øvre trappesum

er øvre trappesum pga. g er voksende

$x \in [0,1]$

og g pos.

c) Def. følgen $[a_n(x)]_{n=1}^{\infty}$ v/

$$a_1(x) = g(x)$$

$$a_2(x) = \int_0^x g(t)\,dt \qquad \left(a_3(x) = \int_0^x \int_0^t g(s)\,ds\,dt \right.$$

$$\cdots$$

$$a_n(x) = \int_0^x a_{n-1}(t)\,dt$$

Vis at følgen konvergerer for hver fast $x \in [0,1]$:

Lås en $x \in [0,1]$. Merk først at $a_n(x) \ge 0\ \forall n$ (fra def. av $a_n(x)$ og samme type argument som i a), evt. bruk av a) generelt). Ved induksjon og bruk av a) kan man vise at a_n er voksende i argumentet, spes. er $a_n(x) \ge a_n(y)$ for alle $y \le x$ og for alle n.

Da er, for alle n:

(groveste øvre trappesum: øvre pga. a_{n-1} voksende i arg

$$a_n(x) = \int_0^x a_{n-1}(t)\,dt \le a_{n-1}(x)(x-0)$$

$$= a_{n-1}(x)\,x \le a_{n-1}(x).$$

$x \in [0,1]$

Dette viser at $[a_n(x)]_n$ er en avtagende følge som er nedre begrenset av 0, og fra Teorem 4.3.9 betyr dette at følgen konvergerer.

8.4: Det ubegrensede integralet.

1.) a) $\int \frac{1}{x+3}\,dx = \underline{\underline{\ln|x+3| + C}}$

b) $\int (7x + 3x^{\frac{1}{2}} - \cos x)\,dx$

$= \underline{\underline{\frac{7}{2}x^2 + 2x^{\frac{3}{2}} - \sin x + C}}$

c) $\int \frac{1}{1+2x^2}\,dx = \int \frac{1}{1+(\sqrt{2}x)^2}\,dx$

$= \underline{\underline{\frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + C}}$

d) $\int \left(8e^{7x} + \underbrace{\frac{1}{\sqrt{x}}}_{=x^{-\frac{1}{2}}}\right)dx = \underline{\underline{\frac{8}{7}e^{7x} + 2x^{\frac{1}{2}} + C}}$

e) $\int \frac{4}{\sqrt{7-x^2}}\,dx = 4\int \frac{1}{\sqrt{7(1-\frac{x^2}{7})}}\,dx$

$= 4\int \frac{1}{\sqrt{7}\sqrt{1-(\frac{x}{\sqrt{7}})^2}}\,dx = \frac{4}{\sqrt{7}}\int \frac{1}{\sqrt{1-(\frac{x}{\sqrt{7}})^2}}\,dx$

$= \frac{4}{\sqrt{7}}\arcsin\left(\frac{x}{\sqrt{7}}\right)\sqrt{7} + C = \underline{\underline{4\arcsin\left(\frac{x}{\sqrt{7}}\right) + C}}$

2.) a) $\int \frac{\quad}{\sin^2(7x)} dx \quad \int \frac{\quad}{\sin^2(7x)}$

$= 42\left(-\cot(7x)\frac{1}{7}\right) + C$

$= \underline{\underline{-6\cot(7x) + C}}$

b) $\int x e^{-x^2} dx = \underline{\underline{-\frac{1}{2} e^{-x^2} + C}}$

c) $\int e^{x} \cos(e^{x}) dx = \underline{\underline{\sin(e^{x}) + C}}$

d) $\int \frac{1}{\sqrt{x}\,\cos^2(\sqrt{x})} dx = \underline{\underline{2\tan(\sqrt{x}) + C}}$

e) $\int \frac{1+x}{1+x^2} dx = \int \frac{1}{1+x^2} dx + \int \frac{x}{1+x^2} dx$

$= \underline{\underline{\arctan x + \frac{1}{2}\ln(1+x^2) + C}}$

3) a) $\int \sqrt{\frac{\arcsin x}{1-x^2}}\, dx = \underline{\underline{\frac{2}{3}(\arcsin x)^{\frac{3}{2}} + C}}$

b) $\int \sin 2x \frac{e^{\cos^2 x}}{e^{\sin^2 x}} dx = \int \sin 2x\, e^{\cos^2 x - \sin^2 x} dx$

$= \int \sin 2x\, e^{\cos 2x} dx = \underline{\underline{-\frac{1}{2} e^{\cos 2x} + C}}$

c) $\int \frac{1}{\sqrt{x}(1+x)} dx = \int \frac{1}{\sqrt{x}(1+(\sqrt{x})^2)} dx$

$= 2\int \frac{1}{2\sqrt{x}(1+(\sqrt{x})^2)} dx = \underline{\underline{2\arctan(\sqrt{x}) + C}}$

d) $\int \frac{7x-1}{\sqrt{1-x^2}}dx = \int \frac{7x}{\sqrt{1-x^2}}dx - \int \frac{1}{\sqrt{1-x^2}}dx$

$= 7\int \frac{x}{\sqrt{1-x^2}}dx - \arcsin x + C$

$= \underline{\underline{-7\sqrt{1-x^2} - \arcsin x + C}}$

5.) $f: (0,\infty) \to \mathbb{R}, \quad f(xy) = f(x) + f(y) \quad \forall\, x, y \in (0,\infty) \quad (\star)$

f er deriverbar i $x=1$ m/ $f'(1) = k$.

a) Vis $f(1)=0$: Velg $x=y=1$:

$f(xy) = f(1\cdot 1) = f(1) \overset{(\star)}{=} f(x) + f(y) = f(1) + f(1) = 2f(1)$.

Så: $f(1) = 2f(1)$

$\underline{0 = f(1)}$

b) Vis $f(x+h) = f(x) + f(1+\frac{h}{x})$:

$f(x) + f(1+\frac{h}{x}) \overset{(\star)}{=} f(x(1+\frac{h}{x})) = f(x+h)$ $\boxtimes$

Bruk dette til å vise at $f'(x) = \frac{k}{x}$:

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \overset{\boxed{\text{over}}}{=} \lim_{h\to 0} \frac{f(x)+f(1+\frac{h}{x})-f(x)}{h}$

Def. av derivert

$= \lim_{h\to 0} \frac{f(1+\frac{h}{x})}{h}$

$$= \lim_{h \to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{xh}{x}} = \frac{1}{x} \lim_{h \to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

x er konstant her

⇓

kan tas utenfor grense

$$= \frac{1}{x} \lim_{y \to 0} \frac{f(1+y) - 0}{y} \overset{a)}{=} \frac{1}{x} \lim_{y \to 0} \frac{f(1+y) - f(1)}{y}$$

La $y := \frac{h}{x}$

$h \to 0 \Rightarrow$

$y = \frac{h}{x} \to 0$

$$= \frac{1}{x} f'(1) = \underline{\underline{\frac{k}{x}}}$$

Def. av den deriverte

c) $f'(x) = \frac{k}{x}$

⇓ (Analysens fundamental-teorem)

$$f(x) = k \ln x + C$$

Men $\underset{\overset{a)}{=} 0}{f(1)} = k \ln(1) + C = 0 + C = C$

over

⇓

$C = 0$

Så: $f(x) = \underline{\underline{k \ln x}}$.