Plenum 9/11-12

9.1: Delvis integrasjon

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

(utledes fra produktregelen for derivasjon)

1.) a) $\int x \sin x\,dx = -x\cos x + \int \cos x\,dx$

$u(x) = x$
$v'(x) = \sin x$
$\Downarrow$
$v(x) = -\cos x$
(Tar m/C helt til slutt)

$= \underline{\underline{\sin x - x\cos x + C}}$

b) $\int x \ln x\,dx = \frac{x^2 \ln x}{2} - \int \frac{1}{\cancel{x}}\frac{1}{2}x^{\cancel{2}}dx$

$u(x) = \ln x$
$v'(x) = x$
$\Downarrow$
$v(x) = \frac{1}{2}x^2$
$u'(x) = \frac{1}{x}$

$= \frac{x^2 \ln x}{2} - \frac{1}{2}\frac{x^2}{2} + C$

$= \underline{\underline{\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C}}$

e) $\int \arctan x\,dx = \int 1 \cdot \arctan x\,dx$

$= x\arctan x - \int x \frac{1}{1+x^2}dx$

$u(x) = \arctan x$
$v'(x) = 1$
$\Downarrow$
$u'(x) = \frac{1}{1+x^2}$
$v(x) = x$

$= \underline{\underline{x\arctan - \frac{1}{2}\ln(1+x^2) + C}}$

f) $\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx$

$= x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx$

$u(x) = \arcsin x$

$v'(x) = 1$

$\Downarrow$

$u'(x) = \frac{1}{\sqrt{1-x^2}}$

$v(x) = x$

$= x \arcsin x + \sqrt{1-x^2} + C$

5.) $\int \frac{\ln(x^2)}{x^2} dx = -\frac{\ln(x^2)}{x} - \int \frac{2}{x} \left(-\frac{1}{x}\right) dx$

$u(x) = \ln(x^2)$

$v'(x) = \frac{1}{x^2} = x^{-2}$

$\Downarrow$

$u'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$

$v(x) = -x^{-1} = \frac{-1}{x}$

$= -\frac{\ln(x^2)}{x} + 2 \int x^{-2} dx$

$= -\frac{\ln(x^2)}{x} + 2(-x^{-1}) + C = -\frac{\ln(x^2)}{x} - \frac{2}{x} + C$

$= -\frac{2}{x}\left(\ln(x) + 1\right) + C$

9.) $\int \sin(\ln x)\,dx = \int 1 \cdot \sin(\ln x)\,dx$

$= x\sin(\ln x) - \int x\cos(\ln x)\frac{1}{x}\,dx$

$u(x) = \sin(\ln x)$
$v'(x) = 1$
$\Downarrow$
$u'(x) = \cos(\ln x)\frac{1}{x}$
$v(x) = x$

$= x\sin(\ln x) - \int \cos(\ln x)\,dx + C$

<u>M:</u> $\int \cos(\ln(x))\,dx$

$= \int 1 \cdot \cos(\ln x)\,dx$

$= x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}\,dx$

$= x\cos(\ln x) + \int \sin(\ln x)\,dx$

$\Downarrow$

$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx + C$

$\Downarrow$

$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) +$

$\int \sin(\ln x)\,dx = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + C$

11.) $\int \frac{x^2 \arctan x}{1+x^2}\,dx = (x - \arctan x)\arctan x - \int \frac{x - \arctan x}{1+x^2}\,dx$

$u(x) = \arctan x$
$v'(x) = \frac{x^2}{1+x^2}$
$\Downarrow$
$u'(x) = \frac{1}{1+x^2}$
$v(x) = x - \arctan x$

<u>M:</u> $\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx$

$= x - \arctan x + D$

$$= x\arctan x - \arctan^2 x - \int \frac{x}{1+x^2}\,dx + \int \frac{\arctan x}{1+x^2}\,dx$$

$$= x\arctan x - \arctan^2 x - \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\arctan^2 x + C$$

$$= x\arctan x - \frac{1}{2}\arctan^2 x - \frac{1}{2}\ln(1+x^2) + C$$

9.2 : Substitusjon

1.) b) $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx = \int \frac{u}{1+u^2}\,2u\,du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\Downarrow$

$2u\,du = dx$

$$= \int \frac{2u^2}{1+u^2}\,du = 2\int \frac{(1+u^2)-1}{1+u^2}\,du$$

$$= 2\left(\int 1\,du - \int \frac{1}{1+u^2}\,du\right)$$

$$= 2(u - \arctan u) + C$$

$$= 2(\sqrt{x} - \underline{\underline{\arctan}}\sqrt{x}) + C$$

d) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{\cancel{u}}{\sqrt{1-u^2}}\,\frac{1}{\cancel{u}}\,du$

$u = e^x$

$du = e^x\,dx$

$\frac{1}{u}\,du = dx$

$$= \arcsin u + C$$

$$= \arcsin(e^x) + C$$

g) $\int \cos(\ln x)\,dx = \int e^u \cos(u)\,du$

$u = \ln x \Rightarrow x = e^u$

$du = \frac{1}{x}dx$

$x\,du = dx$

$e^u du = dx$

$= e^u \cos(u) + \int e^u \sin u\,du$

Delvis int.

$v' = e^u$

$w = \cos u$

$\Downarrow$

$w' = -\sin u$

$v = e^u$

Igjen!

$= e^u \cos u + e^u \sin u - \int e^u \cos u\,du$

$2\int e^u \cos u\,du = e^u(\cos u + \sin u)$

$\int e^u \cos u\,du = \frac{e^u}{2}(\cos u + \sin u) + C$

$\Downarrow$

$\int \cos(\ln x)\,dx = \frac{e^{\ln x}}{2}(\cos(\ln x) + \sin(\ln x) + C)$

$= \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$

h) $\int \arcsin\sqrt{x}\,dx = \int \arcsin(u)\,2u\,du$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$du = \frac{1}{2u}dx$

$2u\,du = dx$

$= 2\int u \arcsin u\,du = 2\left(\frac{1}{2}u^2 \arcsin u - \int \frac{1}{2}u^2 \frac{1}{\sqrt{1-u^2}}du\right)$

$v'(u) = u$

$w(u) = \arcsin u$

$\Downarrow$

$v(u) = \frac{1}{2}u^2$

$w'(u) = \frac{1}{\sqrt{1-u^2}}$

M: $\int \frac{-u^2}{\sqrt{1-u^2}}\,du = \int \frac{1-u^2-1}{\sqrt{1-u^2}}\,du$

$= \int \frac{1-u^2}{\sqrt{1-u^2}}\,du - \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \int \sqrt{1-u^2}\,du - \arcsin u$

$= \int \sqrt{1-(\sin v)^2}\,\cos v\,dv - \arcsin u$

$u = \sin v$
$du = \cos v\,dv$

$= \int \sqrt{\cos^2 v}\,\cos v\,dv - \arcsin u + C$

$= \int \cos^2 v\,dv - \arcsin u + C$

M2: $\int \cos^2 v\,dv = \int \frac{\cos 2v + 1}{2}\,dv = \frac{1}{2}\left[\frac{1}{2}\sin 2v + v\right] + C$

$= \frac{1}{4}\sin 2v + \frac{1}{2}v + C$

M: $\int \frac{-u^2}{\sqrt{1-u^2}}\,du = \frac{1}{4}\sin(2\arcsin u) + \frac{1}{2}\arcsin u$

$- \arcsin u + C$

$= \frac{1}{4}\sin(2\arcsin u) - \frac{1}{2}\arcsin u + C$

Så: $\int \arcsin\sqrt{x}\,dx = u^2 \arcsin u + \frac{1}{4}\sin(2\arcsin u)$

$\boxed{u = \sqrt{x}}$ $\quad - \frac{1}{2}\arcsin u + C$

$= x\arcsin\sqrt{x} + \frac{1}{4}\sin(2\arcsin\sqrt{x})$

$- \frac{1}{2}\arcsin\sqrt{x} + C$

NB: Får ikke samme som fasit, men tror dette stemmer!

3.) a) $\int_0^{\sqrt{2}} x e^{x^2} dx = \int_0^2 \frac{1}{2} e^u du = \frac{1}{2}[e^u]_{u=0}^2$

$= \frac{1}{2}(e^2 - 1)$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2x} du = dx$
$x = 0 \Rightarrow u = 0$
$x = \sqrt{2} \Rightarrow u = 2$

Kan på kladd:
$\int_0^2 e^u \cancel{x} \frac{1}{2\cancel{x}} d\cancel{x}$

b) $\int_1^e \frac{\ln x}{x} dx = \int_0^1 u\,du = [\frac{1}{2}u^2]_{u=0}^1$

$= \frac{1}{2}$

$u = \ln x$
$du = \frac{1}{x} dx$
$x\,du = dx$
$\int \frac{\ln x}{x} dx = \int \frac{u}{\cancel{x}} \cancel{x} du$
$= \int u\,du$
Grenser: $x = 1 \Rightarrow u = \ln 1 = 0$
$x = e \Rightarrow u = 1$

c) $\int_4^9 \frac{\sqrt{x}+1}{1-\sqrt{x}} dx = \int_2^3 \frac{u+1}{1-u} 2u\,du$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}} dx$
$2u\,du = dx$
$x = 4 \Rightarrow u = 2$
$x = 9 \Rightarrow u = 3$

$= -2 \int_2^3 \frac{u^2+1}{u-1} du = -2 \int_2^3 (u + 2 + \frac{2}{u-1}) du$

M:...
polynomdiv.

$= -2[\frac{1}{2}u^2 + 2u + 2\ln|u-1|]$

$= -2[\frac{1}{2}9 + 2 \cdot 3 + 2\ln 2$
$- \frac{1}{\cancel{2}} \cancel{2^2} - 4 - \cancel{2\ln 1}]$

$$= -2\left(\frac{9}{2} + \cancel{6} + 2\ln 2 - \cancel{6}\right) = \underline{\underline{-(9 + 4\ln 2)}}$$

M: Polynomdiv:

$$u^2 + u : u - 1 = \underline{\underline{u + 2 + \frac{2}{u-1}}}$$

$$\begin{array}{l} -(u^2 - u) \\ \hline 2u \\ -(2u - 2) \\ \hline 2 \end{array}$$

d) $$\int_0^3 \arctan\sqrt{x}\,dx = \int_0^{\sqrt{3}} \arctan(u)\,2u\,du$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2u\,du = dx$

$x = 0 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \sqrt{3}$

$$= 2\int_0^{\sqrt{3}} \arctan(u)\,u\,du$$

$$= 2\left(\left[\frac{1}{2}u^2 \arctan u\right]_{u=0}^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{\frac{1}{2}u^2}{1+u^2}du\right)$$

$v = \arctan u$

$w' = u$

$v' = \frac{1}{1+u^2}$

$w = \frac{1}{2}u^2$

$$= 3\arctan\sqrt{3} - \int_0^{\sqrt{3}} \frac{1+u^2-1}{1+u^2}du$$

$$= 3\arctan\sqrt{3} - \int_0^{\sqrt{3}} 1\,du + \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{1+u^2}du$$

$$= 3\arctan\sqrt{3} - \sqrt{3} + \frac{1}{2}\left[\arctan u\right]_0^{\sqrt{3}}$$

$$= 3\arctan\sqrt{3} - \sqrt{3} + \arctan\sqrt{3}$$

$$= 4\arctan\sqrt{3} - \sqrt{3}$$

$$= \underline{\underline{\frac{4\pi}{3} - \sqrt{3}}}$$

7.) $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{\sqrt{1+u}}{\cancel{u}}\,2\cancel{u}\,du$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2u\,du = dx$

$= 2\int \sqrt{1+u}\,du = 2\cdot\frac{2}{3}(1+u)^{\frac{3}{2}} + C = \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}}$

9.) $\int_0^1 e^{\arcsin x}\,dx = \int_0^{\frac{\pi}{2}} e^{u}\cos u\,du$

$x=\sin u$

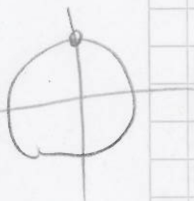
$dx=\cos u\,du$

$= \left[\frac{e^{u}}{2}(\cos(u)+\sin(u))\right]_{u=0}^{\frac{\pi}{2}}$

$\boxed{19}$

$= \frac{1}{2}e^{\frac{\pi}{2}}\left(\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)\right)$

$- \frac{1}{2}(\cos 0 + \sin 0)$



$= \frac{1}{2}e^{\frac{\pi}{2}}(0+1) - \frac{1}{2}(1+0)$

$= \frac{1}{2}e^{\frac{\pi}{2}} - \frac{1}{2}$

15.) $\int_0^{\sqrt{3}} \frac{1+x}{\sqrt{4-x^2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx + \int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,dx$

$= \int_0^{\sqrt{3}} \frac{1}{\sqrt{4(1-(\frac{x}{2})^2)}}\,dx + \int_4^1 \left(-\frac{1}{2}\right)\frac{1}{\sqrt{u}}\,du$

$u = 4-x^2$

$du = -2x\,dx$

$\int \frac{x}{\sqrt{4-x^2}}\,dx$

$= \int \frac{\cancel{x}}{\sqrt{u}}\left(-\frac{1}{2\cancel{x}}\right)du$

$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx + \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

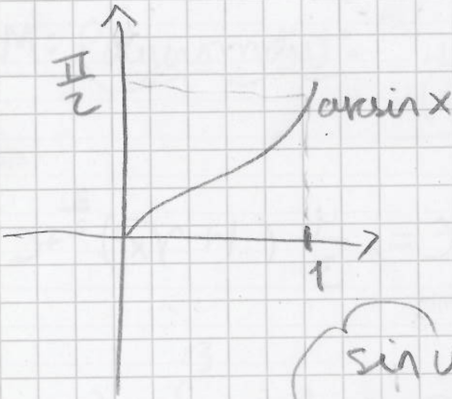
$= \frac{1}{2}\left[\arcsin\left(\frac{x}{2}\right)2\right]_{x=0}^{x=\sqrt{3}} + \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{x}$

$x=0 \Rightarrow u=4$

$x=\sqrt{3} \Rightarrow u=1$

$= \arcsin\frac{\sqrt{3}}{2} - \arcsin 0 + 2 - 1 = \frac{\pi}{3} - 0 + 1 = \frac{\pi}{3} + 1$

23.) $y = \arcsin x$, $0 \le x \le 1$: Finn volum til omdreiningslegemet om x-aksen:



$$V = \int_0^1 \pi (\arcsin x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} u^2 \cos u \, du$$

$\sin u = x$
$\cos u \, du = dx$

$x = 1 \Rightarrow u = \frac{\pi}{2}$
$x = 0 \Rightarrow u = 0$

<u>M:</u> $\int_0^{\frac{\pi}{2}} u^2 \cos u \, du$

$$= \left[u^2 \sin u\right]_{u=0}^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} u \sin u \, du$$

$w(u) = u^2$
$v'(u) = \cos u$
$\Downarrow$
$w'(u) = 2u$
$v(u) = \sin u$

$$= \left[u^2 \sin u\right]_{u=0}^{\frac{\pi}{2}} - 2\left(\left[-u\cos u\right]_{u=0}^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos u \, du\right)$$

$\tilde{w}(u) = u$
$\tilde{v}'(u) = \sin u$
$\Downarrow$
$\tilde{v}(u) = -\cos u$
$\tilde{w}'(u) = 1$

$$= \left[u^2 \sin u + 2u\cos u + 2\sin u\right]_{u=0}^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{4} \cdot 1 + 2\frac{\pi}{2} \cdot 0 + 1(-2)$$

$$- 0 + 0 - 0$$

$$= \underline{\frac{\pi^2}{4} - 2}$$

Dermed er:

$$V = \pi \int_0^{\frac{\pi}{2}} u^2 \cos u \, du = \underline{\underline{\frac{\pi^3}{4} - 2\pi}}$$

25.) $I_n := \int_0^{\frac{\pi}{4}} \tan^n x \, dx \quad n = 0, 1, \ldots$

a) $I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x \, dx = \int_0^{\frac{\pi}{4}} 1 \, dx = [x]_{x=0}^{\frac{\pi}{4}} = \underline{\underline{\frac{\pi}{4}}}$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx$

$= \int_1^{\frac{\sqrt{2}}{2}} \frac{-1}{u} du = [\ln u]_{u=\frac{\sqrt{2}}{2}}^{1} = 0 - \ln \frac{\sqrt{2}}{2}$

$u = \cos x$
$du = -\sin x \, dx$
$\int \frac{\sin x}{\cos x} dx$
$= \int \frac{\cancel{\sin x}}{u} \left(-\frac{1}{\cancel{\sin x}}\right) dx$

$= -\int_{\frac{\sqrt{2}}{2}}^{1} -\frac{1}{u} du$
$= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{u} du$

$= -(\ln \sqrt{2} - \ln 2)$
$= \ln 2 - \ln \sqrt{2}$
$= \ln \left(\frac{2}{\sqrt{2}}\right)$
$= \ln \sqrt{2}$
$= \ln 2^{\frac{1}{2}} = \underline{\underline{\frac{1}{2} \ln 2}}$

$$\boxed{\begin{aligned} x = 0 &\Rightarrow u = 1 \\ x = \tfrac{\pi}{4} &\Rightarrow u = \tfrac{\sqrt{2}}{2} \end{aligned}}$$

b) Vis: $\tan^{n+2} x = \tan^n x \left(\frac{1}{\cos^2 x} - 1\right)$:

$\tan^n x \left(\frac{1}{\cos^2 x} - 1\right) = \tan^n x \left(\frac{1 - \cos^2 x}{\cos^2 x}\right)$

$= \tan^n x \left(\frac{\sin^2 x}{\cos^2 x}\right) = \tan^n x \left(\frac{\sin x}{\cos x}\right)^2$

$= \tan^n x (\tan x)^2 = \underline{\underline{\tan^{n+2} x}}$ ⊠

Vis: $I_{n+2} = \frac{1}{n+1} - I_n$:

$I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^{n+2} x \, dx = \int_0^{\frac{\pi}{4}} \tan^n x \left(\frac{1}{\cos^2 x} - 1\right) dx$

$= \int_0^{\frac{\pi}{4}} \frac{\tan^n x}{\cos^2 x} dx - \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \frac{\tan^n x}{\cos^2 x} dx -$

M: $\int_0^{\frac{\pi}{4}} \frac{\tan^n x}{\cos^2 x}\,dx = \int_0^1 u^n\,du$

$= \left[\frac{1}{n+1} u^{n+1}\right]_0^1$

$= \frac{1}{n+1}$

$u = \tan x$

$du = \frac{1}{\cos^2 x}\,dx$

$\int \frac{\tan^n x}{\cos^2 x}\,dx$

$= \int \frac{\cancel{\tan}\, u^n}{\cancel{\cos^2 x}} \cancel{\cos^2 x}\,du$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{4} \Rightarrow u = 1$

$\Downarrow$

$$I_{n+2} = \frac{1}{n+1} - I_n$$

$= \boxtimes$

c) Vis v/ induksjon:

$(\star)$: $I_{2n+1} = \frac{(-1)^n}{2}\left[\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n}\right)\right]$

$n=1$: VS: $I_3 = \frac{1}{2} - I_1 = \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$

(b)

HS: $\frac{(-1)^1}{2}\left[\ln 2 - \frac{(-1)^2}{1}\right] = \frac{1}{2}(1 - \ln 2)$

Så OK for $n=1$.

Anta $(\star)$ er OK for $n-1$. Vil vise OK for n.

$I_{2n+1} = \frac{1}{2n} - I_{2n-1} = \frac{1}{2n} - I_{2(n-1)+1}$

$= \frac{1}{2n} - \frac{(-1)^{n-1}}{2}\left[\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^n}{n-1}\right)\right]$

$$= \frac{(-1)^n}{2}\left[\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^n}{n-1} + \frac{(-1)^{n+1}}{n}\right)\right]$$

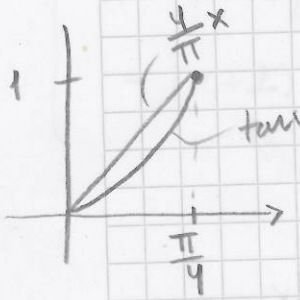
Dermed er påstanden OK v/ induksjon.

d) Forklar hvorfor $\lim_{n\to\infty} \int_0^{\frac{\pi}{4}} \tan^n x \, dx = 0$ og vis at

$$\ln 2 = \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n}\right):$$

For $x \in [0, \frac{\pi}{4}]$ er $\tan x \geq 0 \Rightarrow \tan^n x \geq 0 \Rightarrow$
$0 \leq \int_0^{\frac{\pi}{4}} \tan^n x \, dx$.

Dessuten, for $x \in [0, \frac{\pi}{4}]$ er $\tan x \leq \frac{4}{\pi} x \Rightarrow$



$$0 \leq \int_0^{\frac{\pi}{4}} \tan^n x \, dx \leq \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi} x\right)^n dx$$

$$= \left(\frac{4}{\pi}\right)^n \int_0^{\frac{\pi}{4}} x^n dx = \left(\frac{4}{\pi}\right)^n \left[\frac{1}{n+1} x^{n+1}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n+1} \left(\frac{4}{\cancel{\pi}}\right)^{\cancel{n}} \left(\frac{\cancel{\pi}}{\cancel{4}}\right)^{\cancel{n+1}} = \frac{1}{n+1} \frac{\pi}{4} = \frac{\pi}{4(n+1)}$$

Men: $\lim_{n\to\infty} \frac{\pi}{4(n+1)} = 0 \Rightarrow \lim_{n\to\infty} \int_0^{\frac{\pi}{4}} \tan^n x \, dx = 0$,

siden $0 \leq \int_0^{\frac{\pi}{4}} \tan^n x \, dx \leq \frac{\pi}{4(n+1)}$

Siden $\lim_{n\to\infty} \int_0^{\frac{\pi}{4}} \tan^n x \, dx = 0$, må $\lim_{n\to\infty} I_{2n+1} = 0$

$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{2}\left[\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n}\right)\right] = 0$$

$$\Downarrow$$

$$\ln 2 = \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n}\right)$$

$\boxtimes$

9.3 : Delbrøkoppspalting

1.) d) $\int \frac{x+7}{x^2-x-2}\,dx$

$$x^2-x-2=0$$

$$x=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm 3}{2}=\begin{cases}2\\-1\end{cases}$$

$$\frac{x+7}{x^2-x-2}=\frac{x+7}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$$

$$x+7=A(x+1)+B(x-2)$$
$$=x(A+B)+(A-2B)$$

$$A+B=1 \Rightarrow A=1-B$$
$$A-2B=7 \Rightarrow 1-3B=7$$
$$-3B=6 \Rightarrow B=-2$$
$$A=1-(-2)=3$$

Så:

$$\int\frac{x+7}{x^2-x-2}\,dx=\int\frac{3}{x-2}\,dx-\int\frac{2}{x+1}\,dx$$

$$=\underline{\underline{3\ln|x-2|-2\ln|x+1|+C}}$$

3.) a) $\int\frac{2}{x^2+6x+10}\,dx$

Fullst. kvadrat i nevner:

$$x^2+6x+10=x^2+6x+\left(\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+10$$
$$=(x+3)^2+1$$

$$\int\frac{2}{x^2+6x+10}\,dx=2\int\frac{1}{(x+3)^2+1}\,dx$$

$$= 2\int \frac{1}{u^2+1}\,du = 2\arctan u + C$$

$u = x+3$
$du = dx$

$$= 2\arctan(x+3) + C$$

b) $$\int \frac{2x-2}{x^2+4x+8}\,dx = \int \frac{2x+4}{x^2+4x+8}\,dx - 6\int \frac{dx}{x^2+4x+8}$$

Nemner kan ikke faktoriseres!

$$= \int \frac{1}{u}\,du - 6\int \frac{dx}{(x+2)^2+4}$$

$u = x^2+4x+8$
$du = (2x+4)dx$

$$= \ln|u| - \frac{6}{4}\int \frac{dx}{\left(\frac{x+2}{2}\right)^2+1}$$

$x^2+4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+8$
$= (x+2)^2+4$

$$= \ln|x^2+4x+8| - \frac{3}{2}2\int \frac{1}{u^2+1}\,du$$

$u = \frac{x+2}{2}$
$du = \frac{1}{2}dx$

$$= \ln(x^2+4x+8) - 3\arctan\left(\frac{x+2}{2}\right) + C$$

c) $$\int \frac{x+4}{x^2+4x+3}\,dx = \int \frac{x+4}{(x+1)(x+3)}\,dx$$

$x^2+4x+3=0$
$x = \frac{-4\pm\sqrt{16-4\cdot 3}}{2}$
$= \frac{-4\pm\sqrt{16-12}}{2}$
$= \frac{-4\pm 2}{2} = -2\pm 1 = \begin{cases} -1 \\ -3 \end{cases}$

$$x+4 = A(x+1) + B(x+3) = x(A+B) + (3B+A)$$

$$1 = A+B \Rightarrow A = 1-B$$
$$4 = 3B+A \Rightarrow 4 = 3B+1-B$$

$$3 = 2B \Rightarrow B = \frac{3}{2}$$
$$A = 1-\frac{3}{2} = -\frac{1}{2}$$

$$\int \frac{x+4}{x^2+4x+3}\,dx = -\frac{1}{2}\int \frac{1}{x+3}\,dx + \frac{3}{2}\int \frac{1}{x+1}\,dx$$

$$= -\frac{1}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$$

5.) a) $\int \frac{x^2+2x-3}{x+1}\,dx$

Polynom-div:

$$x^2+2x-3 : x+1 = x+1 - \frac{4}{x+1}$$
$$\frac{-(x^2+x)}{x-3}$$
$$\frac{-(x+1)}{-4}$$

$$\int \frac{x^2+2x-3}{x+1}\,dx = \int \left(x+1 - \frac{4}{x+1}\right)dx$$

$$= \frac{1}{2}x^2 + x - 4\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}x^2 + x - 4\ln|x+1| + C$$

f) $\int \frac{3x^2+x}{(x-1)(x+1)^2}\,dx$

Delbrøkoppsp:

$$\frac{3x^2+x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}$$

$$3x^2+x = A(x+1)^2 + B_1(x-1)(x+1) + B_2(x-1)$$
$$= A(x^2+2x+1) + B_1(x^2-1) + B_2(x-1)$$
$$= x^2(A+B_1) + x(2A+B_2) + (A-B_1-B_2)$$
$$\Downarrow$$

$A+B_1=3$, $2A+B_2=1$, $A-B_1-B_2=0$

$B_1 = 3-A$ $\quad$ $B_2 = 1-2A$ $\quad$ $A-3+A-1+2A=0$

$4A=4$

$B_1 = 2$, $B_2 = -1$ $\Leftarrow$ $A=1$

$$\int \frac{3x^2+x}{(x-1)(x+1)^2}\,dx = \int \frac{1}{x-1}\,dx + 2\int \frac{1}{x+1}\,dx$$

$$-\int \frac{1}{(x+1)^2}\,dx$$

$$= \ln|x-1| + 2\ln|x+1| - \int \frac{1}{u^2}\,du$$

$u = x+1$
$du = dx$

$$= \ln|x-1| + 2\ln|x+1| + u^{-1} + C$$

$$= \ln|x-1| + 2\ln|x+1| + \frac{1}{x+1} + C$$

g) $\displaystyle\int \frac{-x^2+2x-1}{(x+1)(x^2+1)}\,dx$

Delb. oppsp: $\displaystyle\frac{-x^2+2x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$$-x^2+2x-1 = A(x^2+1) + (Bx+C)(x+1)$$

$$= x^2(A+B) + x(B+C) + (A-C)$$

$A+B=-1$, $B+C=2$, $A-C=-1$

$B=-1-A$, $-1-A+C=2$

$$\Downarrow \qquad\qquad \Downarrow$$

$C = 3+A \Rightarrow A+3+A=-1$

$2A=-4$

$B=-1+2=\underline{1} \Leftarrow C=3-2=\underline{1} \Leftarrow A=\underline{-2}$

$$\int \frac{-x^2+2x-1}{(x+1)(x^2+1)}\,dx = -2\int \frac{1}{x+1}\,dx + \int \frac{x+1}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= -2\ln|x+1| + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \arctan x$$

$u = x^2+1$
$du = 2x\,dx$

$$= -2\ln|x+1| + \frac{1}{2}\int \frac{1}{u}\,du + \arctan x$$

$$= -2\ln|x+1| + \frac{1}{2}\ln(x^2+1) + \arctan x + C$$

9) $\int \frac{x+1}{(x-1)(x^2+x+1)}\,dx$

$x^2+x+1=0 \to x = \frac{1-\sqrt{1-4}}{2}$: Kan ikke faktoriseres mer (reelt)

Delb. oppsp.: $\frac{x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$$x+1 = A(x^2+x+1) + (Bx+C)(x-1)$$
$$= x^2(A+B) + x(A-B+C) + (A-C)$$

$A+B=0$, $A-B+C=1$, $A-C=1$

$A=-B$ $\quad -2B+C=1 \quad \Downarrow$

$C=1+2B \Rightarrow -B-1-2B=1$

$-3B=2$

$\underline{A=\frac{2}{3}} \Leftarrow C = 1-\frac{4}{3} = \underline{-\frac{1}{3}} \Leftarrow \underline{B=-\frac{2}{3}}$

$$\int \frac{x+1}{(x-1)(x^2+x+1)}\,dx = \frac{2}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{3}\int \frac{1}{u}\,du$$

$\boxed{u=x^2+x+1 \quad du=(2x+1)dx}$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{3}\ln(x^2+x+1) + C$$

$$= \frac{1}{3}\ln\left[\frac{(x-1)^2}{x^2+x+1}\right] + C$$

17.) $\int \frac{dx}{x^3+8} = \int \frac{dx}{(x+2)(x^2-2x+4)}$

$x^3+8 : x+2 = x^2-2x+4$
$-(x^3+2x^2)$
$-2x^2+8$
$-(-2x^2-4x)$
$4x+8$
$-(4x+8)$
0

Dbos:

$$\frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

$$1 = A(x^2-2x+4) + (Bx+C)(x+2)$$
$$= x^2(A+B) + x(-2A+2B+C) + (4A+2C)$$

$A + B = 0$, $-2A + 2B + C = 0$, $4A + 2C = 1$

$\Downarrow$

$A = -B \Rightarrow 2B + 2B + C = 0$

$A = \frac{1}{12}$ $\quad C = -4B \Rightarrow -4B - 8B = 1$

$-12B = 1$

$\Downarrow$

$C = \frac{4}{12} = \frac{1}{3} \Leftarrow$ $\quad B = -\frac{1}{12}$

$$\int \frac{dx}{x^3+8} = \frac{1}{12}\int \frac{1}{x+2}\,dx + \int \frac{-\frac{1}{12}x + \frac{1}{3}}{x^2-2x+4}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int \frac{2x-8}{x^2-2x+4}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int \frac{2x-2}{x^2-2x+4}\,dx + \frac{6}{24}\int \frac{dx}{x^2-2x+4}$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int \frac{1}{u}\,du + \frac{1}{4}\int \frac{dx}{(x-1)^2+3}$$

$\boxed{\begin{array}{l} u = x^2-2x+4 \\ du = 2x-2dx \end{array}}$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2-2x+4)$$

$$+ \frac{1}{12}\int \frac{dx}{\left(\frac{x-1}{\sqrt{3}}\right)^2+1}$$

x^2-2x+4

$= x^2-2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+4$

$= x^2-2x+1+3$

$= (x-1)^2+3$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2-2x+4)$$

$$+ \frac{\sqrt{3}}{12}\arctan\left(\frac{x-1}{\sqrt{3}}\right) + C$$

21.) a) $$\int \underbrace{\frac{u+2}{u^2+2u+5}}\,du = \frac{1}{2}\int \frac{2u+4}{u^2+2u+5}\,du$$

an ikke faktoriseres reelt

$$= \frac{1}{2}\int \frac{2u+2}{u^2+2u+5}\,du + \frac{2}{2}\int \frac{1}{u^2+2u+5}\,du$$

$$= \frac{1}{2}\int \frac{1}{v}\,dv + \int \frac{1}{(u+1)^2+4}\,du$$

$v = u^2+2u+5$
$dv = (2u+2)\,du$

$$= \frac{1}{2}\ln(u^2+2u+5) + \frac{1}{4}\int \frac{1}{\left(\frac{u+1}{2}\right)^2+1}\,du$$

$$u^2+2u+5 = u^2+2u+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+5$$
$$= u^2+2u+1+4$$
$$= (u+1)^2+4$$

$$= \frac{1}{2}\ln(u^2+2u+5) + \frac{2}{4}\arctan\left(\frac{u+1}{2}\right) + C$$

$$= \frac{1}{2}\ln(u^2+2u+5) + \frac{1}{2}\arctan\left(\frac{u+1}{2}\right) + C$$

b) $$\frac{1}{u(u^2+2u+5)} = \frac{A}{u} + \frac{Bu+C}{u^2+2u+5}$$

$$1 = A(u^2+2u+5) + (Bu+C)u$$
$$= u^2(A+B) + u(2A+C) + 5A$$

$A+B=0$, $2A+C=0$, $5A=1$
$\Downarrow$
$B = -\frac{1}{5}$, $-\frac{2}{5} = C \Leftarrow A = \frac{1}{5}$

c) Regn ut:

$$\int \frac{\tan x}{\cos^2 x+2\cos x+5}\,dx = \int \frac{\sin x}{\cos x(\cos^2 x+2\cos x+5)}\,dx$$

$$= -\int \frac{1}{u(u^2+2u+5)}\,du \underset{b)}{=} -\frac{1}{5}\int \frac{1}{u}\,du + \frac{1}{5}\int \frac{u+2}{u^2+2u+5}\,du$$

$u = \cos x$
$du = -\sin x\,dx$

$$\underset{a)}{=} -\frac{1}{5}\ln|u| + \frac{1}{10}\ln(u^2+2u+5) + \frac{1}{10}\arctan\left(\frac{u+1}{2}\right) + C$$

$$= \frac{1}{10}\ln(\cos^2 x + 2\cos x + 5) + \frac{1}{10}\arctan\left(\frac{\cos x + 1}{2}\right)$$

(Tror fortegnsfeil i fasit)

$$- \frac{1}{5}\ln|\cos x| + C$$

25.) a) 2+i er rot i $z^3 - 11z + 20 = 0$ fordi:

$$(2+i)^3 - 11(2+i) + 20 = (4 + 4i - 1)(2+i)$$
$$- 22 - 11i + 20 = \cancel{8} + \cancel{4i} + \cancel{8i} - \cancel{4} - \cancel{2} - \cancel{i} - \cancel{2}$$
$$= 0$$

Reelt polynom $\Rightarrow$ Røttene kommer i kompleks-konjugerte par $\Rightarrow$ 2−i er rot og:

$$z^3 - 11z + 20 = (z - (2-i))(z - (2+i))(z - ?)$$

$$(z - (2-i))(z - (2+i)) = z^2 - z(2+i) - (2-i)z + (2-i)(2+i)$$
$$= z^2 - z(2 + \cancel{i} + 2 - \cancel{i}) + 5$$
$$= z^2 - 4z + 5$$

$$z^3 - 11z + 20 : z^2 - 4z + 5 = z + 4$$
$$-(z^3 - 4z^2 + 5z)$$
$$4z^2 - 16z + 20$$
$$-(4z^2 - 16z + 20)$$
$$0$$

$\Downarrow$

Røttene er $2+i$, $2-i$ og

b) $\displaystyle\int \frac{10x+3}{x^3-11x+20}\,dx = \int \frac{10x+3}{(x+4)(x^2-4x+5)}\,dx$

a)

Dbos: $\displaystyle\frac{10x+3}{(x+4)(x^2-4x+5)} = \frac{A}{x+4} + \frac{Bx+C}{x^2-4x+5}$

$$10x+3 = A(x^2-4x+5) + (Bx+C)(x+4)$$
$$= x^2(A+B) + x(-4A+4B+C) + (5A+4C)$$

$A+B=0$, $-4A+4B+C=10$, $3=5A+4C$

$A=-B$, $8B+C=10$

$C=10-8B$, $3=-5B+40-32B$

$37B=37$

$\underline{A=-1}$, $\underline{C=2}$ $\Leftarrow$ $\underline{B=1}$

$$\int \frac{10x+3}{x^3-11x+20}\,dx = -\int \frac{1}{x+4}\,dx + \int \frac{x+2}{x^2-4x+5}\,dx$$

$$= -\ln|x+4| + \frac{1}{2}\int \frac{2x-4}{x^2-4x+5}\,dx + 4\int \frac{1}{x^2-4x+5}\,dx$$

$$= -\ln|x+4| + \frac{1}{2}\int \frac{1}{u}\,du + 4\int \frac{1}{(x-2)^2+1}\,dx$$

$\boxed{u = x^2+4x+5 \quad du = 2x\,dx}$

$$= -\ln|x+4| + \frac{1}{2}\ln(x^2-4x+5) + 4\arctan(x-2) + C$$

$x^2-4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+5$
$=(x-2)^2+1$

27.) $$\int \frac{1}{e^{2x}+4e^{x}+13}\,dx = \int \frac{1}{u(u^2+4u+13)}$$

$u = e^{x}$
$du = e^{x}dx$

Kan ikke faktorisere mer.

Obs:
$$\frac{1}{u(u^2+4u+13)} = \frac{A}{u} + \frac{Bu+C}{u^2+4u+13}$$

$$1 = A(u^2+4u+13) + (Bu+C)u$$
$$= u^2(A+B) + u(4A+C) + 13A$$

$A+B=0, \quad 4A+C=0, \quad 13A=1$

$$\Downarrow$$

$\underline{B=-\frac{1}{13}}, \quad \underline{C=-\frac{4}{13}}, \quad \underline{A=\frac{1}{13}}$

$$\int \frac{1}{e^{2x}+4e^{x}+13}\,dx = \frac{1}{13}\int \frac{1}{u}\,du - \frac{1}{13}\int \frac{u+4}{u^2+4u+13}\,du$$

$$= \frac{1}{13}\ln|u| - \frac{1}{26}\int \frac{2u+4}{u^2+4u+13}\,du$$
$$- \frac{2}{13}\int \frac{1}{u^2+4u+13}\,du$$

$$= \frac{1}{13}\ln|u| - \frac{1}{26}\int \frac{1}{v}\,dv$$

$v = u^2+4u+13$
$dv = (2u+4)\,du$

$$- \frac{2}{13}\int \frac{1}{(u+2)^2+9}\,du$$

$u^2+4u+13$
$= u^2+4u+4-4+13$
$= (u+2)^2+9$

$$= \frac{1}{13}\ln|u| - \frac{1}{26}\ln(u^2+4u+13)$$
$$+ \frac{2}{13\cdot 9}\int \frac{1}{\left(\frac{u+2}{3}\right)^2+1}\,du$$

$$= \frac{1}{13}\ln|u| - \frac{1}{26}\ln(u^2+4u+13)$$
$$- \frac{2\cdot 3}{13\cdot 9}\arctan\left(\frac{u+2}{3}\right) + C$$

$$= \frac{1}{13} \ln e^x - \frac{1}{26} \ln(e^{2x} + 4e^x + 13)$$
$$- \frac{2}{39} \arctan\left(\frac{e^x+2}{3}\right) + C$$

$$= \frac{1}{13}\left(x - \frac{1}{2} \ln(e^{2x} + 4e^x + 13) - \frac{2}{3} \arctan\left(\frac{e^x+2}{3}\right)\right) + C$$

31.) $\int \ln(x^2 + 2x + 10)\,dx$

$$= x\ln(x^2+2x+10) - \int \frac{x(2x+2)}{x^2+2x+10}\,dx$$

Delvis int:
$u(x) = \ln(x^2+2x+10)$
$v'(x) = 1$
$\Downarrow$
$u'(x) = \frac{2x+2}{x^2+2x+10}$
$v(x) = x$

$$= x\ln(x^2+2x+10) - \int \frac{2x^2+2x}{x^2+2x+10}\,dx$$

M: $\int \frac{2x^2+2x}{x^2+2x+10}\,dx$

Polynomdiv: $2x^2 + 2x : x^2 + 2x + 10 = 2 - \frac{2x+20}{x^2+2x+10}$
$$\frac{-(2x^2+4x+20)}{-2x-20}$$

$$\int \frac{2x^2+2x}{x^2+2x+10}\,dx = \int 2\,dx - \int \frac{2x+20}{x^2+2x+10}\,dx$$

$$= 2x - \int \frac{2x+2}{x^2+2x+10}\,dx - 18\int \frac{1}{x^2+2x+10}\,dx$$

$$= -\frac{18}{9} \int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx - \int \frac{1}{u}\,du + 2x$$

$u = x^2+2x+10$
$du = (2x+2)dx$

$$= -2\arctan\left(\frac{x+1}{3}\right)\cdot 3 - \ln(x^2+2x+10) + C$$

$$= -6\arctan\left(\frac{x+1}{3}\right) - \ln(x^2+2x+10) + C$$

Så:

$$\int \ln(x^2+2x+10)\,dx = x\ln(x^2+2x+10)$$

$$+ 6\arctan\left(\frac{x+1}{3}\right)$$

$$+ \ln(x^2+2x+10) - 2x +$$

$$= (x+1)\ln(x^2+2x+10)$$

$$\underline{\underline{+ 6\arctan\left(\frac{x+1}{3}\right) - 2x + C}}$$

9.5: Uegentlige integraler

1) a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \arctan x \Big]_{x=0}^{a}$

$$= \lim_{a\to\infty} \arctan a - \arctan 0$$

$$= \frac{\pi}{2} - 0 = \underline{\underline{\frac{\pi}{2}}}$$

b) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = [\arcsin x]_{x=0}^{1} = \arcsin 1 - \arcsin$

$$= \frac{\pi}{2} - 0 = \underline{\underline{\frac{\pi}{2}}}$$

3) a) $\displaystyle\int_0^\infty \frac{x+4}{x^2+2x+1}\,dx$ Avgjør om integralet konvergerer eller divergerer

Sammenligner m/ $f(x) = \frac{1}{x}$ (som divergerer fra Set. 8.

$$\lim_{x\to\infty} \frac{\frac{x+4}{x^2+2x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x^2+4x}{x^2+2x+1}$$

$$= \lim_{x\to\infty} \frac{1+\frac{4}{x}}{1+\frac{2}{x}+\frac{1}{x^2}} = 1 > 0 \Rightarrow \int_0^\infty \frac{x+4}{x^2+2x+1}\,dx \text{ divergere}$$

fra grensesammenligningskrite

c) $\displaystyle\int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$

Merk: $\displaystyle\frac{1}{\sqrt{x+x^3}} \underset{x\in[0,1]}{\leq} \frac{1}{\sqrt{x^3}} = \frac{1}{x^{\frac{3}{2}}}$

Men siden $\displaystyle\int_0^1 \frac{1}{x^{\frac{3}{2}}}\,dx$ konvergerer fra Set. 9.5.8

og $\displaystyle\frac{1}{\sqrt{x+x^3}} \leq \frac{1}{x^{\frac{3}{2}}}$ vil $\displaystyle\int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$ også

konvergere.

6.) Avgjør om $\displaystyle\int_0^1 \ln(x^3+x^2)\,dx$ konvergerer eller divergerer:

$$\int_0^1 \ln(x^3+x^2)\,dx = \lim_{a\to 0^+} \int_a^1 \ln(x^3+x^2)\,dx$$

Merk: $\ln(x^3) < \ln(x^3+x^2) < \ln(2x^2)$

(siden $x\in[0,1]$ og ln voksende)

NB: $\displaystyle\int_a^1 \ln(x^3)\,dx \underset{\text{Delvis integrasjon}}{=} 3\,[x\ln x - x]_{x=a}^{1}$

$= -3 - a\ln a + a$

M: $\displaystyle\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} \overset{\text{"}\frac{-\infty}{\infty}\text{": L'Hôpital}}{=} \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}}$

$\displaystyle= \lim_{a\to 0^+} a = \underline{0}$

Dermed er: $\displaystyle\lim_{a\to 0} \int_a^1 \ln(x^3)\,dx = \underline{-3}$

Siden $\ln(x^3) < \ln(x^3+x^2) < 0$ (for $x \in [0,1]$) og $\int_0^1 \ln(x^3)\,dx$ konvergerer, gir sammenligningskriteriet at $\int_0^1 \ln(x^3+x^2)\,dx$ <u>konvergerer</u>.

10.) For hvilke p konvergerer

$$\int_0^{\frac{1}{2}} \frac{dx}{x|\ln x|^p} \quad \text{og} \quad \int_2^{\infty} \frac{dx}{x|\ln x|^p} \; ?$$

$$\int_{x=0}^{\frac{1}{2}} \frac{dx}{x|\ln x|^p} = \int_{-\infty}^{-\ln 2} \frac{1}{|u|^p}\,du$$

$u = \ln x$
$du = \frac{1}{x}dx$
$x\,du = dx$

$x = 0 \Rightarrow u = -\infty$
$x = \frac{1}{2} \Rightarrow u = \ln\frac{1}{2} = -\ln 2$

$$\underset{\text{symmetri}}{=} \int_{\ln 2}^{\infty} \frac{1}{u^p}\,du = \underbrace{\int_{\ln 2}^{1} \frac{1}{u^p}\,du}_{\text{Konvergerer } \forall p} + \underbrace{\int_1^{\infty} \frac{1}{u^p}\,du}_{\text{Set. 9.5.4: Konvergerer for } p>1 \text{, divergerer for } p\leq 1}$$

Konvergerer for $p>1$, divergerer for $p \leq 1$.

Så: $\int_0^{\frac{1}{2}} \frac{dx}{x|\ln x|^p}$ konvergerer for $p>1$ og divergerer <u>for $p \leq 1$.</u>

$$\int_2^{\infty} \frac{dx}{x|\ln x|^p} \overset{!}{=} \int_{\ln 2}^{\infty} \frac{1}{u^p}\, du$$

$u = \ln x$
$du = \frac{1}{x} dx$

$x \overset{!}{=} 2 \Rightarrow u = \ln 2$
$x = \infty \Rightarrow u = \infty$

$\Downarrow$ (samme arg. som over)

$\int_2^{\infty} \frac{dx}{x|\ln x|^p}$ konvergerer for $p > 1$ og divergerer for $p \leq 1$.